$|f(x)|$ and $f(|x|)$ Are Not the Same Graph

Mathematics | JEE Main & Advanced

Don't Memorize. Visualize.

The same function and the same modulus give two different graphs, because the order matters. In $|f(x)|$ the modulus is outside: it acts on the output y, so every part of the graph below the x-axis turns over it. In $f(|x|)$ the modulus is inside: it acts on the input x, so the left half is thrown away and the right half is mirrored in the y-axis.

MODULUS OUTSIDE

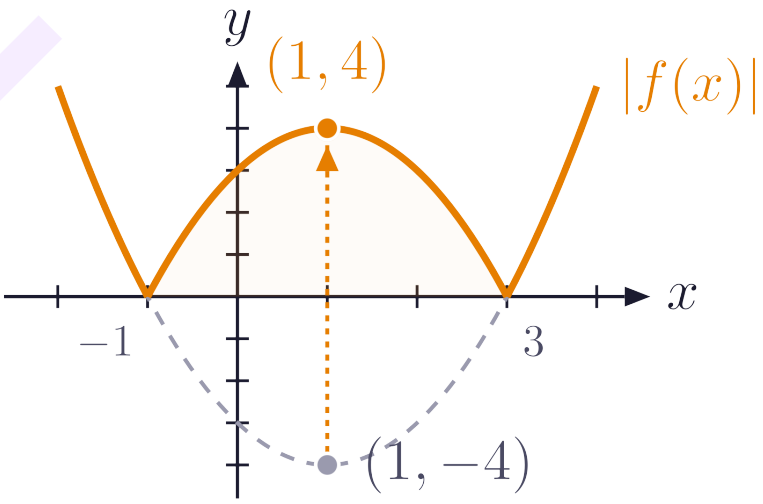
$|f(x)|$: first find $f(x)$, then take its modulus.
A height $y \geq 0$ stays; a height $y < 0$ becomes $-y$. The graph never goes below the x-axis.

MODULUS INSIDE

$f(|x|)$: first take $|x|$, then put it into f.
For $x \geq 0$ nothing changes. At $x = -a$ the graph takes the height of f at $+a$, so it is symmetric about the y-axis.

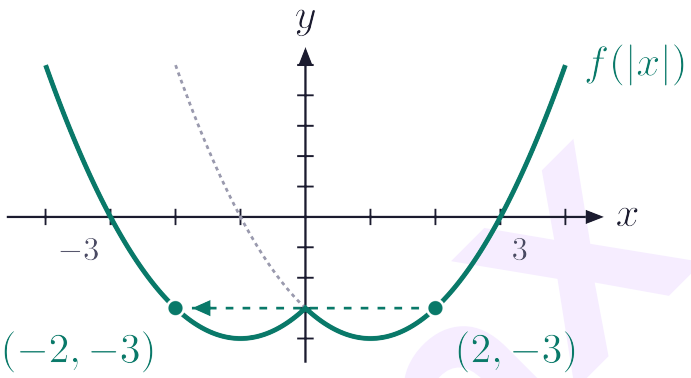
Modulus outside: $|f(x)|$ turns over the x-axis

Take $f(x) = x^2 - 2x - 3 = (x+1)(x-3)$. Its graph cuts the x-axis at -1 and 3, lies below the x-axis between them, and its lowest point is $(1, -4)$.
At $x = 1$, $|f(1)| = |-4| = 4$, so the point $(1, -4)$ goes to $(1, 4)$. The same happens to every point below the x-axis: the whole part from -1 to 3 turns over the x-axis, like a page. The parts already above the x-axis do not change.



Modulus inside: $f(|x|)$ mirrors the right half

For $x \geq 0$, $|x| = x$, so $f(|x|) = f(x)$: the right half stays.
For a negative x, say $x = -2$: $f(|-2|) = f(2) = -3$. The point at $x = -2$ takes the height of f at $x = 2$, so $(2, -3)$ is copied to $(-2, -3)$. The old value $f(-2) = 5$ is never used. The whole right half is mirrored in the y-axis, and the old left half is thrown away.



CAREFUL

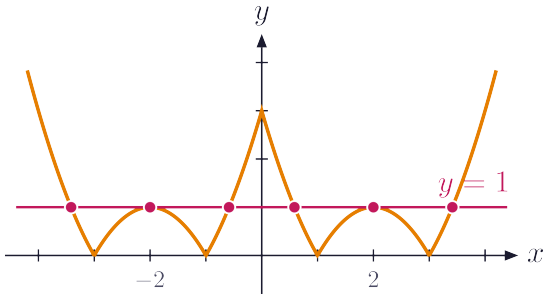
Never keep the old left part of f when you draw $f(|x|)$. Here f has a root at -1, but $f(|-1|) = f(1) = -4$, not 0: that root is lost. The roots of $f(|x|)$ are $\pm a$ for each root $a \geq 0$ of f; here only 3, so the roots are -3 and 3.

Solved example

Find the number of real solutions of $|x^2 - 4|x| + 3| = 1$.

Write it with g. $x^2 = |x|^2$, so $x^2 - 4|x| + 3 = g(|x|)$ with $g(x) = x^2 - 4x + 3 = (x-1)(x-3)$.

Inside first, then outside. Keep the right half of g (roots 1 and 3, lowest point $(2, -1)$) and mirror it in the y-axis. Then turn every part below the x-axis upwards: the two dips become humps of height 1, with tops at $x = -2$ and $x = 2$.



Count. The line $y = 1$ touches the two tops and cuts the graph at four more points. By algebra: $g(|x|) = 1$ gives $|x| = 2 \pm \sqrt{2}$ (four values of x), and $g(|x|) = -1$ gives $|x| = 2$ (two values).

six real solutions

Key formulas

$$|f(x)| = \begin{cases} f(x), & f(x) \geq 0 \\ -f(x), & f(x) < 0 \end{cases} \qquad f(|x|) = \begin{cases} f(x), & x \geq 0 \\ f(-x), & x < 0 \end{cases}$$

$|f(x)| \geq 0$ for every x. $f(|x|)$ is symmetric about the y-axis: $f(|-x|) = f(|x|)$.

Practice set

Easy to hard. For each, ask first: is the modulus outside or inside? The answers start on the next page.

Q1. Quiz from the Short. $f(x) = (x+2)(x-4)$. What are the roots of $f(|x|)$?
(A) -2 and 4 (B) -4 and 4 (C) $-4, -2, 2, 4$.

Q2. For $f(x) = x^2 - 2x - 3$, find $|f(1)|$ and $f(|-1|)$.

Q3. How many times does the graph of $y = |x^2 - 2x - 3|$ meet the line $y = 2$?

Q4. How many real roots does $x^2 - 2|x| - 3 = 0$ have?

Q5. For $f(x) = x^2 - 2x - 3$, find every x for which $|f(x)| = f(|x|)$.

Q6. How many real solutions does $|x^2 - 4|x| + 3| = \frac{1}{2}$ have?

Watch the animated lesson on YouTube
$|f(x)|$ and $f(|x|)$ are not the same graph (Short)
Scan the code to open the JEEVisionX channel on YouTube. This lesson is in the JEE Concepts Visualization Shorts playlist.

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Answers

Q1

(B) -4 and 4

$f(|x|) = 0$ means $|x| = -2$ or $|x| = 4$. $|x| = -2$ has no solution, so $x = -4$ or 4. (A) keeps the roots of f itself: -2 belongs to the old left part, which is thrown away. (C) mirrors both roots, but $|x|$ is never -2.

Q2

$|f(1)| = 4$, $f(|-1|) = -4$

$f(1) = 1 - 2 - 3 = -4$, so $|f(1)| = 4$. $|-1| = 1$, so $f(|-1|) = f(1) = -4$: outside the modulus acts on the value, inside it acts on x.

Q3

4 times

$|f(x)| = 2$ means $f(x) = 2$ or $f(x) = -2$. $x^2 - 2x - 5 = 0$ gives $x = 1 \pm \sqrt{6}$; $x^2 - 2x - 1 = 0$ gives $x = 1 \pm \sqrt{2}$, both between -1 and 3, on the turned hump.

Q4

2 roots

$x^2 - 2|x| - 3 = f(|x|)$ with $f(x) = x^2 - 2x - 3 = (x+1)(x-3)$. So $|x| = 3$ or $|x| = -1$; only $|x| = 3$ works: $x = -3$ and $x = 3$.

Q5

every $x \geq 3$

For $x \geq 0$, $f(|x|) = f(x)$, and $|f(x)| = f(x)$ only where $f(x) \geq 0$: $x \geq 3$. For $x < -1$, $|f(x)| = x^2 - 2x - 3$ but $f(|x|) = x^2 + 2x - 3$; they differ by $4x \neq 0$. For $-1 \leq x < 0$, $|f(x)| \geq 0 > f(|x|)$, since $f(|x|) = f(-x)$ and $0 < -x \leq 1$ is inside the dip.

Q6

8 solutions

$g(|x|) = \frac{1}{2}$ gives $|x| = 2 \pm \sqrt{\frac{3}{2}}$ and $g(|x|) = -\frac{1}{2}$ gives $|x| = 2 \pm \sqrt{\frac{1}{2}}$, where $g(x) = x^2 - 4x + 3$. All four values are positive, and each gives two values of x.