Why Is $x + \frac{1}{x}$ at Least 2?

Maths | JEE Main & Advanced

Don't Memorize. Visualize.

For every positive x, $x + \frac{1}{x} \geq 2$, and it is equal to 2 only at $x = 1$. The reason is one picture: four rectangles of area 1 round a square of side $x + \frac{1}{x}$ leave a square hole in the middle, so the big square's area is at least 4.

FIXED PRODUCT

x and $\frac{1}{x}$ multiply to 1 whatever x is: a rectangle with these sides always has area 1.

$x \times \frac{1}{x} = 1$

LEAST VALUE

The smallest value the expression takes. Here it is 2, reached at $x = 1$ only.

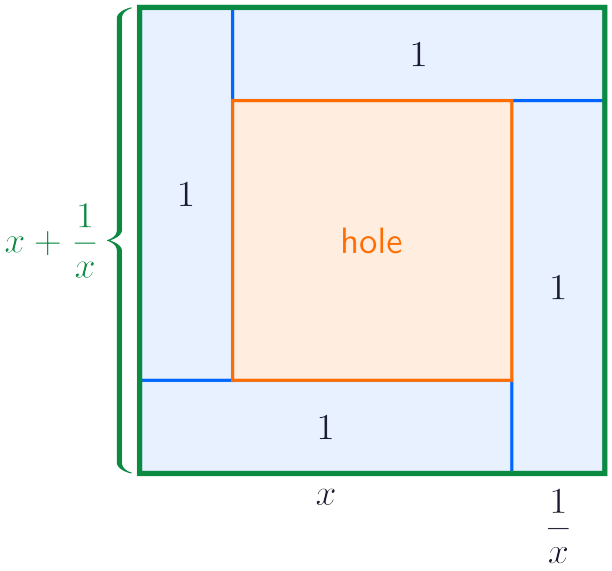
$\min\left(x + \frac{1}{x}\right) = 2$

The picture

Take $x = 2$. A rectangle with sides $x = 2$ and $\frac{1}{x} = \frac{1}{2}$ has area 1.

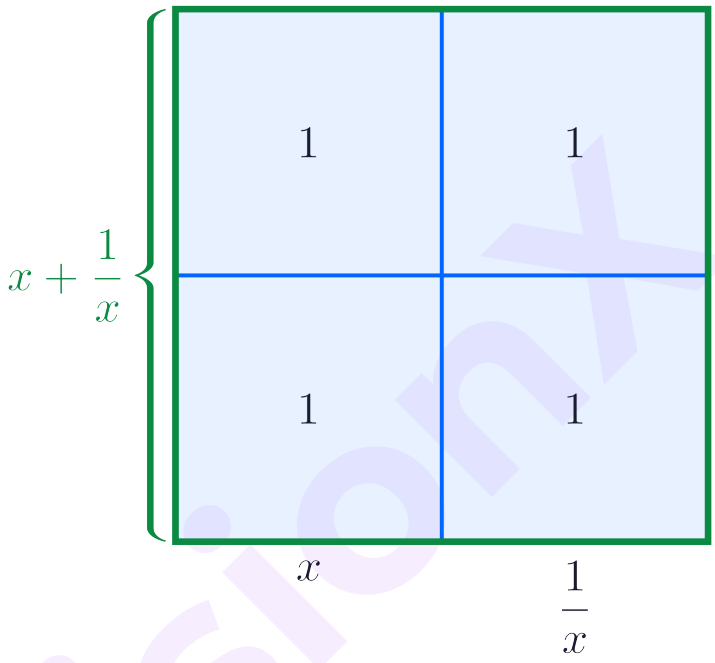
Four such rectangles, each turned a quarter turn from the last, fit round a square of side $x + \frac{1}{x} = 2.5$. In the middle they leave a square hole of side $x - \frac{1}{x} = 1.5$.

So the big square's area is 4×1 plus the hole: $\left(x + \frac{1}{x}\right)^2 = 4 + \left(x - \frac{1}{x}\right)^2 \geq 4$. The side is positive, so $x + \frac{1}{x} \geq 2$.



When is it exactly 2?

The sum is 2 only when the hole has no size, that is, when $x = \frac{1}{x}$. For positive x this means $x = 1$.
Then the four rectangles are unit squares, and they fill a square of side exactly 2.
For any other positive x there is a hole, and the sum is more than 2.



CAREFUL

x **must be positive.** For negative x the sum is negative: $x + \frac{1}{x} \leq -2$, equal to -2 at $x = -1$. So $x + \frac{1}{x}$ has no least value over all $x \neq 0$.

The least value must be reached. The same idea gives $t + \frac{1}{t} \geq 2$ for any positive t, but 2 is the least value only if $t = 1$ is allowed.

Solved example

Find the least value of $x + \frac{4}{x}$ for $x > 0$.

Product. $x \times \frac{4}{x} = 4$: four rectangles of area 4 round a square of side $x + \frac{4}{x}$, so $\left(x + \frac{4}{x}\right)^2 \geq 16$ and $x + \frac{4}{x} \geq 4$.

When. No hole when $x = \frac{4}{x}$, that is, at $x = 2$: $2 + 2 = 4$.

4, at $x = 2$

Key formulas

$$\left(x + \frac{1}{x}\right)^2 = 4 + \left(x - \frac{1}{x}\right)^2$$

$$x + \frac{1}{x} \geq 2 \quad \text{for } x > 0, \text{ equal only at } x = 1$$

$$a + b \geq 2\sqrt{ab} \quad \text{for } a, b > 0, \text{ equal only when } a = b$$

Practice set

Easy to hard. The answers start on the next page.

Q1. Quiz from the Short. What is the least value of $\tan\theta + \cot\theta$, for $0 < \theta < \frac{\pi}{2}$? (A) 1 (B) 2 (C) 4.

Q2. Find the least value of $4^x + 4^{-x}$, for real x.

Q3. Find the least value of $x + \frac{9}{x}$ for $x > 0$, and the x where it is reached.

Q4. For $x < 0$, find the greatest value of $x + \frac{1}{x}$.

Q5. Find the least value of $\sin^2\theta + \operatorname{cosec}^2\theta$.

Q6. Find the least value of $\frac{x^2+2}{\sqrt{x^2+1}}$, for real x.

Watch the animated lesson on YouTube

Why Is $x + \frac{1}{x}$ at Least 2? (Short)

Scan the code to open the JEEVisionX channel on YouTube. This lesson is in the JEE Concepts Visualization Shorts playlist.

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Answers

Q1

(B) 2

On $\left(0, \frac{\pi}{2}\right)$, $\tan\theta > 0$ and $\cot\theta = \frac{1}{\tan\theta}$, so the sum is $t + \frac{1}{t} \geq 2$, equal at $\tan\theta = 1$, $\theta = \frac{\pi}{4}$. 1 is the product; 4 is the square of the sum.

Q2

2

$t = 4^x > 0$ and $4^{-x} = \frac{1}{t}$: $t + \frac{1}{t} \geq 2$, equal at $t = 1$, $x = 0$.

Q3

6, **at** $x = 3$

The product is 9: $x + \frac{9}{x} \geq 2\sqrt{9} = 6$, equal when $x = \frac{9}{x}$, $x = 3$.

Q4

-2

Put $x = -t$ with $t > 0$:
$x + \frac{1}{x} = -\left(t + \frac{1}{t}\right) \leq -2$, equal at $t = 1$, $x = -1$.

Q5

2

$t = \sin^2\theta$ is in $(0, 1]$ and $\operatorname{cosec}^2\theta = \frac{1}{t}$:
$t + \frac{1}{t} \geq 2$, equal at $t = 1$ (allowed: $\theta = \frac{\pi}{2}$).

Q6

2

$\frac{x^2+2}{\sqrt{x^2+1}} = \sqrt{x^2+1} + \frac{1}{\sqrt{x^2+1}} \geq 2$, equal when $\sqrt{x^2+1} = 1$, at $x = 0$.

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