Greatest $|PA - PB|$ for P on a Line

Maths | JEE Main & Advanced

Don't Memorize. Visualize.

Let A and B be on the same side of a line, and let P move on the line. Then $|PA - PB|$ is never more than AB, and it equals AB at the point where the line AB meets the line P moves on. So the greatest value of $|PA - PB|$ is AB.

THE TRIANGLE RULE

In any triangle, each side is less than the sum of the other two: $PB < PA + AB$.

So $PB - PA < AB$.

WHEN IT IS EQUAL

When P, A and B are on one line with A between P and B, the triangle is flat: $PB = PA + AB$.

Then $PB - PA = AB$ exactly.

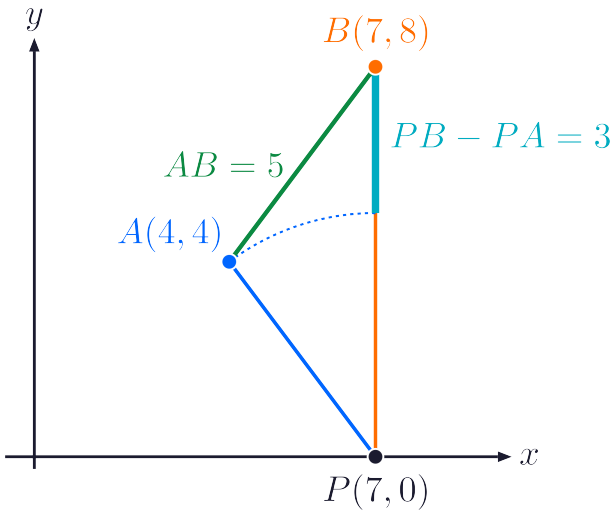
Why it is true: turn PA onto PB

Take $A(4,4)$, $B(7,8)$ and $P(7,0)$ on the x-axis. Here $PA = 5$ and $PB = 8$.

Turn PA about P until it lies along PB. The piece of PB left over is $PB - PA = 3$.

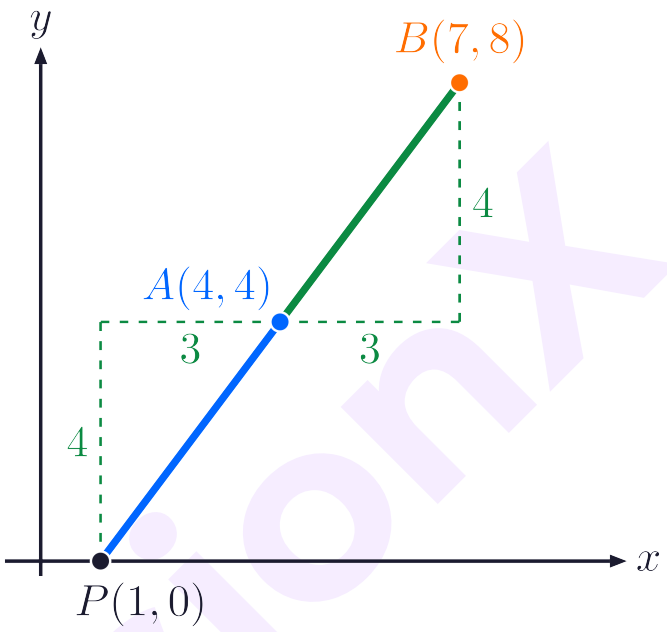
The path from P to A to B is longer than the straight PB, so $PA + AB > PB$ and the piece is less than $AB = 5$.

This holds for every P on the axis that is not on the line AB.



Where the greatest value is

Slide P along the axis. The piece grows until P reaches the line AB: there A lies on PB, and the piece is AB itself.
From A to B is 3 right and 4 up, so $AB = \sqrt{3^2+4^2} = 5$. The same step back from A (3 left, 4 down) reaches the axis at $P(1,0)$.
Check: $PA = 5$, $PB = 10$, $PB - PA = 5 = AB$.



CAREFUL

Do not mirror here. For the *least* $PA + PB$ with A, B on the same side, you mirror one point in the line. For the *greatest* $|PA - PB|$ with A, B on the same side, you do not: the answer is AB itself.
Opposite sides. If A and B are on opposite sides of the line, mirror one of them first; then they are on the same side and the rule above applies.
AB **parallel to the line.** Then the line AB never meets it, and $|PA-PB|$ has no greatest value: it comes as close to AB as you like but never reaches it.

Solved example

$A(1,2)$ and $B(7,10)$; P moves on the x-axis. Find the greatest value of $|PA-PB|$ and the point P where it occurs.

Greatest value. A and B are above the axis, so it is $AB = \sqrt{6^2+8^2} = 10$.

Where. The step from A to B is 6 right and 8 up. Going down 2 from A to the axis is $\frac{1}{4}$ of a step back: $x = 1 - \frac{6}{4} = -\frac{1}{2}$. Check: $PA = \frac{5}{2}$, $PB = \frac{25}{2}$, $PB - PA = 10$.

10, at $P\left(-\frac{1}{2}, 0\right)$

Key results

A, B on the same side: greatest $|PA - PB| = AB$, where line AB meets the line

A, B on the same side: least $PA + PB = AB'$, with B' the mirror image of B

A, B on opposite sides: greatest $|PA - PB| = AB'$, with B' the mirror image of B

Practice set

Easy to hard. P moves on the x-axis unless the question says otherwise. The answers start on the next page.

Q1. Quiz from the Short. $A(0,2)$, $B(12,7)$. The greatest value of $|PA-PB|$ is: (A) 15 (B) 13 (C) 5.

Q2. $A(1,1)$, $B(4,5)$. Find the greatest value of $|PA-PB|$ and the point P.

Q3. $A(2,3)$, $B(5,7)$, and P moves on the y-axis. Find the greatest value of $|PA-PB|$ and the point P.

Q4. $A(0,3)$, $B(4,3)$. Does $|PA-PB|$ have a greatest value? Explain.

Q5. $A(1,2)$, $B(5,-4)$. Find the greatest value of $|PA-PB|$ and the point P.

Q6. Find the greatest value of $\left|\sqrt{(x-1)^2+4}-\sqrt{(x-4)^2+16}\right|$ for real x, and the x where it occurs.

Watch the animated lesson on YouTube

Greatest $|PA-PB|$ in One Picture (Short)

Scan the code to open the JEEVisionX channel on YouTube. This lesson is in the JEE Concepts Visualization Shorts playlist.

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Answers

Q1

(B) 13

A and B are on the same side, so the greatest value is $AB = \sqrt{12^2 + 5^2} = 13$, at $P(-4.8, 0)$. 15 is AB' with B mirrored, which is the trick for the least $PA + PB$, not this.

Q2

5, at $P\left(\frac{1}{4}, 0\right)$

$AB = \sqrt{3^2 + 4^2} = 5$. Down 1 to the axis is $\frac{1}{4}$ of the step back: $x = 1 - \frac{3}{4}$.

Q3

5, at $P\left(0, \frac{1}{3}\right)$

Both points have $x > 0$. Left 2 to the axis is $\frac{2}{3}$ of the step back: $y = 3 - \frac{2}{3} \times 4 = \frac{1}{3}$.

Q4

No

AB is parallel to the axis, so P is never on the line AB: $|PA - PB| < 4$ always, and it comes as close to 4 as you like as P moves far away.

Q5

$2\sqrt{5}$, at $P(-3, 0)$

Mirror B to $B'(5, 4)$: $PB' = PB$. Then $AB' = \sqrt{4^2 + 2^2} = 2\sqrt{5}$, and the line AB' meets the axis at $x = -3$.

Q6

$\sqrt{13}$, at $x = -2$

The roots are the distances from $(x, 0)$ to $A(1, 2)$ and $B(4, 4)$: the greatest is $AB = \sqrt{3^2 + 2^2} = \sqrt{13}$, where the line AB meets the axis.