Why $\sin^{-1}(\sin 2)$ Is Not 2

Mathematics | JEE Main & Advanced

Don't Memorize. Visualize.

$\sin^{-1}$ always gives an angle from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$: an angle on the right half of the unit circle. So $\sin^{-1}(\sin x) = x$ only when x itself is in that range. For any other x, the answer is the angle in that range with the same sine. For $x = 2$ it is $\pi - 2 \approx 1.14$.

THE TRAP

"$\sin^{-1}$ undoes $\sin$, so $\sin^{-1}(\sin x) = x$ for every x."

It is true only for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. Here x is in radians, and 2 is outside that range.

PRINCIPAL VALUE

$\sin^{-1} y$ is the one angle θ with $\sin\theta = y$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

On the right half of the circle every height from -1 to 1 comes exactly once, so there is exactly one such angle.

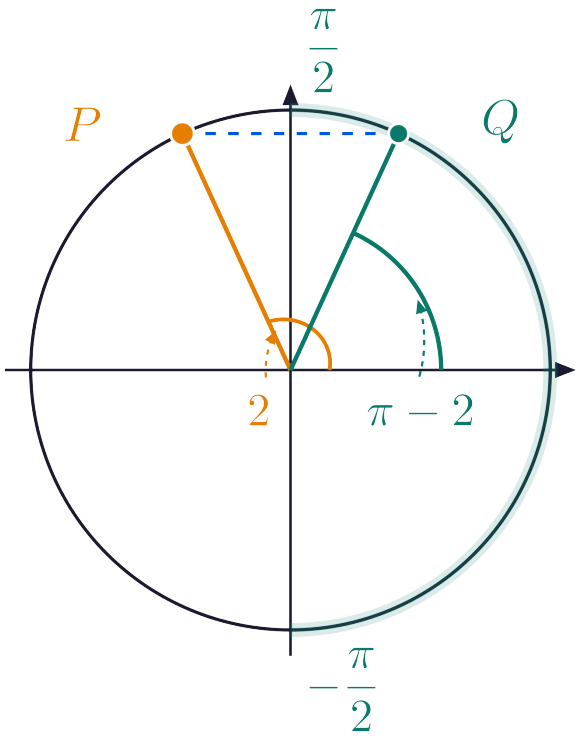
Why the answer is $\pi - 2$

Take the circle of radius 1. The point P is at an angle of 2 radians (about $115°$) from the positive x-axis, and its height is $\sin 2 \approx 0.909$. Since $2 > \frac{\pi}{2}$, P is on the left half, so 2 cannot be the answer.

A half turn is π, so the angle from the radius to P on to the negative x-axis is $\pi - 2$. Take the mirror image in the vertical axis: P lands on Q, at an angle of $\pi - 2$ from the positive x-axis. A mirror image keeps the height, so $\sin(\pi - 2) = \sin 2$.

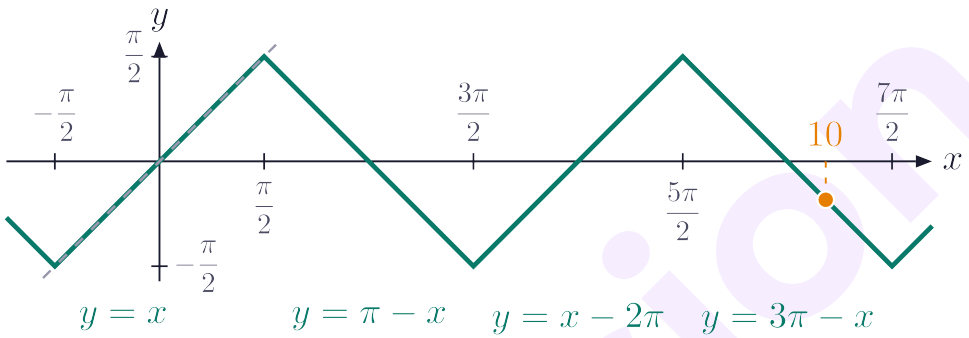
$\pi - 2 \approx 1.14$ lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, so

$$\sin^{-1}(\sin 2) = \pi - 2.$$



The graph of $y = \sin^{-1}(\sin x)$ is a zigzag

Let x grow from 0. The point P goes round the circle, and Q stays on the right half at the same height as P; the angle of Q is $y = \sin^{-1}(\sin x)$. From 0 to $\frac{\pi}{2}$, P is on the right half, so Q is P and $y = x$. From $\frac{\pi}{2}$ to $\frac{3\pi}{2}$, P is on the left half and Q slides down the right half: $y = \pi - x$. From $\frac{3\pi}{2}$ to $\frac{5\pi}{2}$, P is on the right half again, one full turn later: $y = x - 2\pi$. The pattern repeats for every turn, and for negative x too.



Every piece crosses the x-axis at a multiple of π, with slope 1 or -1. The graph meets the dashed line $y = x$ only from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

CAREFUL

Do not cancel $\sin^{-1}$ against $\sin$. The other order is different: $\sin(\sin^{-1} y) = y$ holds for every y from -1 to 1, but $\sin^{-1}(\sin x) = x$ holds only for x from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. Always check that your answer lies in that range.

Solved example

Find $\sin^{-1}(\sin 10)$.

Where is 10? $\frac{5\pi}{2} \approx 7.85$ and $\frac{7\pi}{2} \approx 11.0$, so 10 lies between $\frac{5\pi}{2}$ and $\frac{7\pi}{2}$.

The piece. There the graph crosses the x-axis at 3π and falls with slope -1: $y = 3\pi - x$.

Check. $3\pi - 10 \approx -0.58$, which lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

$\sin^{-1}(\sin 10) = 3\pi - 10$

Key formulas

$\sin^{-1}(\sin x) = x$ exactly when $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ (both ends included). For any other x, find the piece on which x lies and use its line:

$$\sin^{-1}(\sin x) = \begin{cases} -\pi - x, & -\frac{3\pi}{2} \le x \le -\frac{\pi}{2} \\ x, & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \le x \le \frac{3\pi}{2} \\ x - 2\pi, & \frac{3\pi}{2} \le x \le \frac{5\pi}{2} \\ 3\pi - x, & \frac{5\pi}{2} \le x \le \frac{7\pi}{2} \end{cases}$$

Practice set

Easy to hard. For each, find where x lies first. The answers start on the next page.

Q1. Quiz from the Short. What is $\sin^{-1}(\sin(-2))$? (A) -2 (B) $2-\pi$ (C) $\pi-2$.

Q2. Find $\sin^{-1}(\sin 3)$.

Q3. Find $\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$.

Q4. Find $\sin^{-1}(\sin 5)$.

Q5. Find $\sin^{-1}(\sin 7)$.

Q6. Find all x from 0 to 2π for which $\sin^{-1}(\sin x)=\pi-x$.

Watch the animated lesson on YouTube

Why $\sin^{-1}(\sin 2)$ is not 2 (Short)

Scan the code to open the JEEVisionX channel on YouTube. This lesson is in the JEE Concepts Visualization Shorts playlist.

AN INITIATIVE BY AN IIT ALUMNUS

@JEEVisionX

Answers

Q1

(B) $2 - \pi$

-2 lies between $-\frac{3\pi}{2}$ and $-\frac{\pi}{2}$, on the piece $y = -\pi - x$: $-\pi + 2 = 2 - \pi \approx -1.14$. (A) -2 is the trap: it is outside the range. (C) has the wrong sign: $\sin(\pi - 2) = \sin 2$, but $\sin(-2) = -\sin 2$.

Q2

$\pi - 3$

3 lies between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, on the piece $y = \pi - x$. $\pi - 3 \approx 0.14$ is in the range.

Q3

$\frac{\pi}{6}$

$\frac{5\pi}{6}$ lies between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$: $\pi - \frac{5\pi}{6} = \frac{\pi}{6}$.

Q4

$5 - 2\pi$

5 lies between $\frac{3\pi}{2} \approx 4.71$ and $\frac{5\pi}{2} \approx 7.85$, on the piece $y = x - 2\pi$. $5 - 2\pi \approx -1.28$ is in the range.

Q5

$7 - 2\pi$

7 is on the same piece as 5: $7 - 2\pi \approx 0.72$.

Q6

$\frac{\pi}{2} \le x \le \frac{3\pi}{2}$

$y = \pi - x$ is the piece from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$, both ends included. Below $\frac{\pi}{2}$ the value is x, and above $\frac{3\pi}{2}$ it is $x - 2\pi$; neither equals $\pi - x$ there.