Least $PA + PB$: Mirror One Point

Maths | JEE Main & Advanced

Don't Memorize. Visualize.

> A and B are on the same side of a line, and P moves on the line. Mirror B in the line to get B'. Then the least value of $PA + PB$ is AB', and it happens where AB' meets the line.

MIRROR IMAGE

The image of a point in a line: on the other side, at the same distance, with the line as the perpendicular bisector of the join.

In the x-axis, (x, y) goes to $(x, -y)$.

SAME SIDE

Two points are on the same side of $ax + by + c = 0$ when $ax + by + c$ has the same sign at both.

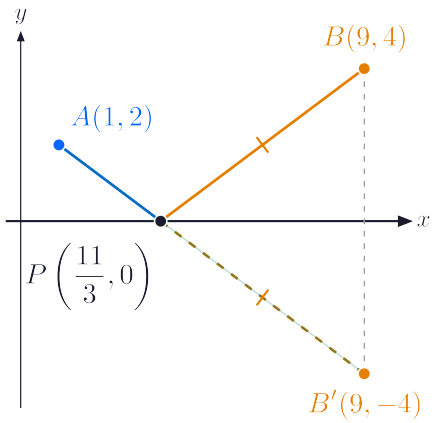
$A(1, 2)$ and $B(9, 4)$: both above the x-axis.

Why the mirror works

P is on the x-axis, $A(1, 2)$, $B(9, 4)$. The mirror image of B in the x-axis is $B'(9, -4)$.

Every point P on the axis is the same distance from B and from B', so $PB = PB'$ (the two equal marks). That turns the question into: when is $PA + PB'$ least?

The path $A \to P \to B'$ crosses the axis, and the shortest such path is the straight line: $AB' = \sqrt{8^2 + 6^2} = 10$, at $P\left(\frac{11}{3}, 0\right)$.



Finding P. Drop perpendiculars from A and from B' to the x-axis. The two right triangles that meet at P have equal angles at P, so they are similar. Their heights are 2 (A above the axis) and 4 (B' below it), so their bases are in the ratio $1 : 2$ as well. The bases add up to $9 - 1 = 8$, so they are $\frac{8}{3}$ and $\frac{16}{3}$, and $x = 1 + \frac{8}{3} = \frac{11}{3}$. The same comes from the section formula: P divides AB' in the ratio $2 : 4$, so $x = \frac{4 \times 1 + 2 \times 9}{2 + 4} = \frac{22}{6} = \frac{11}{3}$.

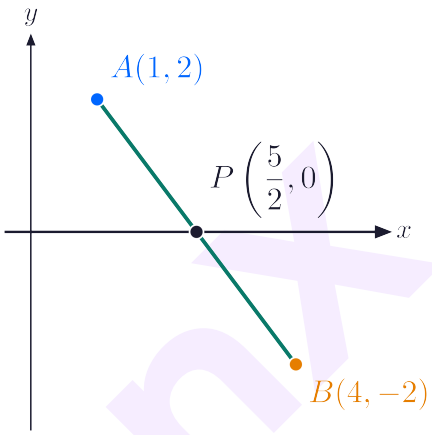
Why the straight line is the least

For any P on the axis, the triangle inequality in triangle APB' gives $PA + PB' \geq AB'$, with equality only when P lies on the segment AB'. Since $PB = PB'$,

$$PA + PB \;\geq\; AB' = 10 \quad \text{for every } P, \qquad \text{equal only at } P = \left(\tfrac{11}{3}, 0\right).$$

When A and B are on opposite sides

No mirror is needed. For $A(1,2)$ and $B(4,-2)$ the straight segment AB already crosses the x-axis, at $P\left(\frac{5}{2},0\right)$. So the least $PA+PB$ is $AB=5$.
Check the side first: put both points into the line's expression. Same sign, mirror one of them; opposite signs, join them directly.



CAREFUL

Not the straight AB when both are on one side. $AB=\sqrt{68}\approx 8.25$ is shorter than 10, but that segment never touches the axis, so no P on the axis gives it.
Not the midpoint. P under the middle of A and B, at $(5,0)$, gives $PA+PB\approx 10.13$. The least is at $x=\frac{11}{3}$, nearer the point closer to the line.
Mirror in the right line. For P on the y-axis, (x,y) goes to $(-x,y)$; for P on $y=x$, (x,y) goes to (y,x).

Solved example

P moves on the y-axis. Find the least value of $PA+PB$ for $A(3,1)$ and $B(5,7)$.

Same side. Both x-coordinates are positive, so mirror A in the y-axis: $A'(-3,1)$.

Least value. $A'B=\sqrt{(5+3)^2+(7-1)^2}=\sqrt{64+36}=10$.

Where. The line $A'B$ has slope $\frac{6}{8}=\frac{3}{4}$, so at $x=0$, $y=1+\frac{3}{4}\times 3=\frac{13}{4}$. Check: $PA=\frac{15}{4}$, $PB=\frac{25}{4}$, and $\frac{15}{4}+\frac{25}{4}=10$.

10, at $P\left(0,\frac{13}{4}\right)$

Key formulas

$$\text{least } PA+PB \text{ (same side)} = AB', \quad B' = \text{image of } B \text{ in the line}$$

$$\text{image of } (x,y) \text{ in the } x\text{-axis: } (x,-y)$$

$$\text{in the } y\text{-axis: } (-x,y)$$

$$\text{in the line } y=x\text{: } (y,x)$$

$$\text{image of } (x_1,y_1) \text{ in } ax+by+c=0: \quad \frac{x-x_1}{a}=\frac{y-y_1}{b}=-\frac{2(ax_1+by_1+c)}{a^2+b^2}$$

Practice set

Easy to hard. P moves on the line named in each question. The answers start on the next page.

Q1. Quiz from the Short. $A(0,1)$, $B(4,2)$, P on the x-axis. The least value of $PA+PB$ is: (A) $\sqrt{17}$ (B) 5 (C) 7.

Q2. $A(1,1)$, $B(4,3)$, P on the x-axis. Find the least value of $PA+PB$ and the point P.

Q3. $A(1,2)$, $B(4,-2)$, P on the x-axis. Find the least value of $PA+PB$ and the point P.

Q4. $A(1,2)$, $B(2,6)$, P on the y-axis. Find the least value of $PA+PB$ and the point P.

Q5. Find the mirror image of $(3,4)$ in the line $y=x$, and in the line $x+y=1$.

Q6. $A(4,1)$, $B(5,2)$, P on the line $y=x$. Find the least value of $PA+PB$ and the point P.

Watch the animated lesson on YouTube

Least PA + PB: Mirror One Point (Short)

Scan the code to open the JEEVisionX channel on YouTube. This lesson is in the JEE Concepts Visualization Shorts playlist.

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Answers

Q1

(B) 5

Both points are above the axis, so mirror B: $B'(4,-2)$. The least value is $AB' = \sqrt{4^2+3^2} = 5$, at $P\left(\frac{4}{3},0\right)$. $\sqrt{17}$ is AB, which never touches the axis; 7 is the length of a path that does touch it (down to the axis, along it, up to B), but not the shortest one.

Q2

5, at $P\left(\frac{7}{4},0\right)$

$B'(4,-3)$: $AB' = \sqrt{3^2+4^2}$; AB' meets $y=0$ at $x = 1+\frac{3}{4}$.

Q3

5, at $P\left(\frac{5}{2},0\right)$

Opposite sides: no mirror. $AB = \sqrt{3^2+4^2}$, and AB crosses the axis at $x = \frac{5}{2}$.

Q4

5, at $P\left(0,\frac{10}{3}\right)$

Mirror A in the y-axis: $A'(-1,2)$, $A'B = \sqrt{3^2+4^2}$; at $x=0$, $y = 2+\frac{4}{3}$.

Q5

$(4,3)$ and $(-3,-2)$

In $y=x$ swap the coordinates. In $x+y-1=0$: $-\frac{2(3+4-1)}{2} = -6$, so $(3-6, 4-6)$. Check: the midpoint $(0,1)$ is on the line.

Q6

$2\sqrt{5}$, at $P(3,3)$

$y-x$ is -3 at both points (same side). $B'(2,5)$, $AB' = \sqrt{2^2+4^2} = 2\sqrt{5}$; AB' meets $y=x$ at $(3,3)$.

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