At Least One Six: Count the Opposite

Mathematics | JEE Main & Advanced

Don't Memorize. Visualize.

To find the probability of *at least one*, find the probability of *none* and subtract it from 1: $P(\text{at least one}) = 1 - P(\text{none})$. Adding the probabilities of the separate events counts the results they share more than once.

COMPLEMENT

The opposite event: every result that is not in A. Written A' or $\bar{A}$.

$P(A) + P(A') = 1$.

INDEPENDENT ROLLS

Rolls that do not affect each other, like the separate dice here.

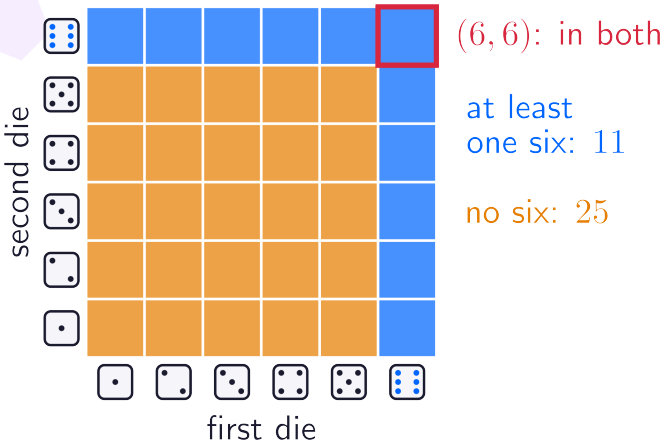
Probability that all of them happen: multiply.

Two dice: why adding goes wrong

Two fair dice have 36 equally likely results, one square each in the grid. A six on the first die is the right column: 6 squares, so $\frac{1}{6}$. A six on the second die is the top row: 6 squares, again $\frac{1}{6}$.

Adding gives $\frac{1}{6} + \frac{1}{6} = \frac{12}{36}$, which counts 12 squares. But the square $(6, 6)$ is in the column *and* in the row, so it is counted twice. Only 11 squares have a six:

$$P(\text{at least one six}) = \frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36}.$$



Count the opposite

"At least one six" fails only when *neither* die shows a six. Then the first die is 1 to 5 and the second die is 1 to 5: the 5×5 block, 25 squares. Every other square has a six.

No six on two dice

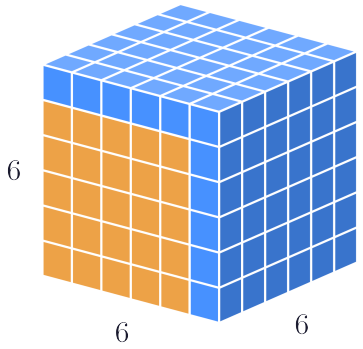
$$P(\text{no six}) = \frac{25}{36} = \left(\frac{5}{6}\right)^2, \qquad P(\text{at least one six}) = 1 - \frac{25}{36} = \frac{11}{36}.$$

The subtraction needs no care about overlaps: the opposite event is one simple block.

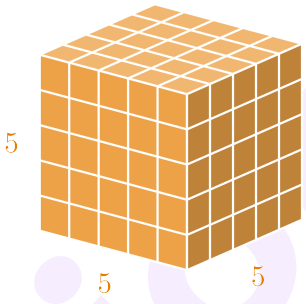
Three dice: the cube

With a third die the 216 results are the small cubes of a $6 \times 6 \times 6$ cube (left). No six on any die is the $5 \times 5 \times 5$ corner cube, 125 results (right), so $P(\text{no six}) = \left(\frac{5}{6}\right)^3$. The rest have at least one six:

$$216 - 125 = 91, \qquad P(\text{at least one six}) = 1 - \left(\frac{5}{6}\right)^3 = \frac{91}{216}.$$



all results: 216



no six: 125

Counted directly, those 91 results split three ways: first die 6 (36); first die not 6 and second die 6 (30); first two not 6 and third die 6 (25). The opposite is one product. For n rolls, $P(\text{at least one six}) = 1 - \left(\frac{5}{6}\right)^n$; for 4 rolls it is $1 - \frac{625}{1296} = \frac{671}{1296} \approx 0.52$. Adding would give $\frac{n}{6}$, a sure six at six rolls: false, because all six rolls can miss.

CAREFUL

"At least one six" is not "exactly one six". Exactly one six in 4 rolls means one roll is a six and three miss, and the six can be any of the 4 rolls:

$$4 \times \frac{1}{6} \times \left(\frac{5}{6}\right)^3 = \frac{500}{1296} = \frac{125}{324} \approx 0.39.$$

Solved example

A coin is tossed 3 times. Find the probability of at least one head.

The opposite. No head means three tails: $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$.

Subtract. $1 - \frac{1}{8} = \frac{7}{8}$.

$\frac{7}{8}$ (adding would give $\frac{3}{2}$, more than 1)

Key formulas

$$P(\text{at least one}) = 1 - P(\text{none})$$

$$\text{independent events: } P(\text{all of them}) = \text{the product of their probabilities}$$

$$\text{probability } p \text{ each, } n \text{ tries: } P(\text{at least one}) = 1 - (1-p)^n$$

$$P(\text{exactly one}) = n\,p\,(1-p)^{n-1}$$

Practice set

Easy to hard. The answers start on the next page.

Q1. **Quiz from the Short.** Two coins are tossed. The probability of at least one head is: (A) 1 (B) $\frac{3}{4}$ (C) $\frac{1}{4}$.

Q2. Two dice are rolled. Find the probability that at least one of them shows a 1 or a 2.

Q3. A die is rolled 4 times. Find the probability of exactly one six.

Q4. What is the least number of rolls of a die for which the probability of at least one six is more than $\frac{1}{2}$?

Q5. A shooter hits a target with probability 0.3 on each shot. Find the probability of at least one hit in 3 shots.

Q6. For the same shooter, find the least number of shots that makes the probability of at least one hit at least 0.9.

Watch the animated lesson on YouTube

At Least One Six (Short)

Scan the code to open the JEEVisionX channel on YouTube. This lesson is in the JEE Concepts Visualization Shorts playlist.

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Answers

Q1

(B) $\frac{3}{4}$

$P(\text{no head}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, so $P(\text{at least one head}) = 1 - \frac{1}{4} = \frac{3}{4}$. Check by listing HH, HT, TH, TT: three of the four have a head. (A) adds $\frac{1}{2} + \frac{1}{2}$ and counts HH twice; (C) is the probability of no head.

Q2

$\frac{5}{9}$

One die shows neither 1 nor 2 with probability $\frac{4}{6}$; both dice: $\left(\frac{4}{6}\right)^2 = \frac{16}{36}$. So $1 - \frac{16}{36} = \frac{20}{36} = \frac{5}{9}$ (20 of the 36 squares).

Q3

$\frac{125}{324}$

$4 \times \frac{1}{6} \times \left(\frac{5}{6}\right)^3 = \frac{500}{1296} = \frac{125}{324}$ (500 of the 1296 results have exactly one six).

Q4

4 rolls

3 rolls: $1 - \frac{125}{216} = \frac{91}{216}$, less than $\frac{1}{2}$. 4 rolls: $\frac{671}{1296}$, more than $\frac{1}{2}$.

Q5

0.657

No hit in 3 shots: $0.7^3 = 0.343$. $1 - 0.343 = 0.657$.

Q6

7 shots

Need $0.7^n \leq 0.1$. $0.7^6 \approx 0.118$ is too big; $0.7^7 \approx 0.082$ is enough.

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