Maximum of $a \sin x + b \cos x$

Mathematics | JEE Main & Advanced

Don't Memorize. Visualize.

The maximum of $a \sin x + b \cos x$ is $\sqrt{a^2 + b^2}$, and its minimum is $-\sqrt{a^2 + b^2}$. For $3 \sin x + 4 \cos x$ that is 5, not $3 + 4 = 7$: the two terms never reach their largest values at the same x.

THE TRAP

$3 \sin x$ is at most 3 and $4 \cos x$ is at most 4, so the sum "must" reach 7.

It would need $\sin x = 1$ and $\cos x = 1$ at the same x. That never happens.

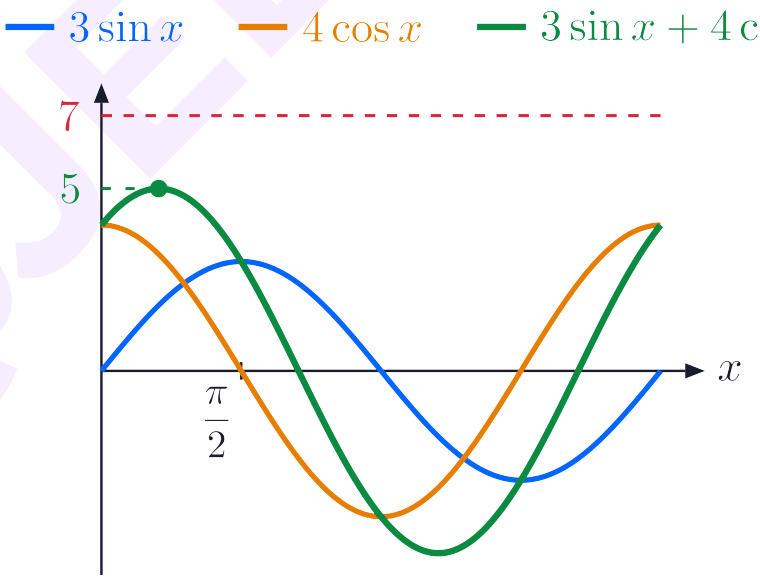
THE RESULT

$-\sqrt{a^2 + b^2} \le a \sin x + b \cos x \le \sqrt{a^2 + b^2}$ for every real x.

Both ends are reached. The signs of a and b do not matter: they are squared.

Why the sum never reaches 7

$3 \sin x$ is largest, 3, at $x = \frac{\pi}{2}$, and there $\cos x = 0$. $4 \cos x$ is largest, 4, at $x = 0$, and there $\sin x = 0$. Add the two at every x: the sum $3 \sin x + 4 \cos x$ is a wave of its own, and its highest value is 5.



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The highest point is where $\tan x = \frac{3}{4}$ ($x \approx 0.64$). There $\sin x = \frac{3}{5}$ and $\cos x = \frac{4}{5}$, so

$$3\sin x + 4\cos x = \tfrac{9}{5} + \tfrac{16}{5} = 1.8 + 3.2 = 5.$$

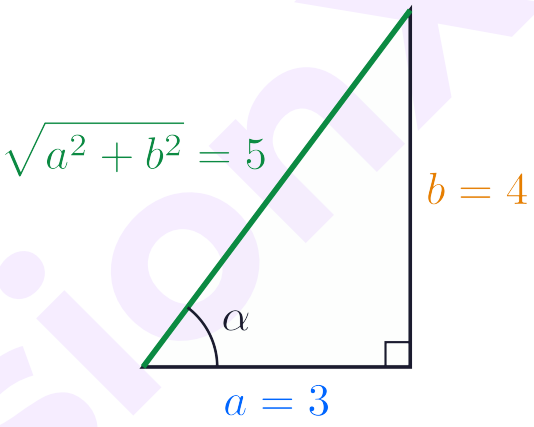
Neither term is at its own maximum there: the best x is a compromise.

Why the maximum is $\sqrt{a^2+b^2}$

Let $R = \sqrt{a^2+b^2}$ (for a and b not both 0), and take the angle α of the right triangle with legs a and b: $\cos\alpha = \frac{a}{R}$, $\sin\alpha = \frac{b}{R}$. Then

$$a\sin x + b\cos x = R(\cos\alpha\sin x + \sin\alpha\cos x) = R\sin(x+\alpha).$$

Since $\sin(x+\alpha)$ runs over every value from -1 to 1, the expression runs from $-R$ to R.
For $3\sin x + 4\cos x$: $R = \sqrt{9+16} = 5$, so $3\sin x + 4\cos x = 5\sin(x+\alpha)$ with $\tan\alpha = \frac{4}{3}$.



The range

$$-\sqrt{a^2+b^2} \;\le\; a\sin x + b\cos x \;\le\; \sqrt{a^2+b^2}$$

Adding a constant moves both ends: $a\sin x + b\cos x + c$ runs from $c - \sqrt{a^2+b^2}$ to $c + \sqrt{a^2+b^2}$.

CAREFUL

Do not add the largest values of the two terms. $3+4=7$ is not the maximum of $3\sin x + 4\cos x$, and $-3-4=-7$ is not its minimum: the equation $3\sin x + 4\cos x = 7$ has no solution at all. The terms must be squared and added under one root: $\sqrt{3^2+4^2} = 5$.

Solved example

Find the greatest and the least value of $6\sin x + 8\cos x + 3$.

The wave. $R = \sqrt{6^2+8^2} = \sqrt{100} = 10$, so $6\sin x + 8\cos x$ runs from -10 to 10.

Add the constant. The whole expression runs from $-10+3$ to $10+3$.

Greatest 13, least -7.

Key formulas

$$a\sin x + b\cos x = \sqrt{a^2+b^2}\,\sin(x+\alpha), \qquad \cos\alpha = \frac{a}{\sqrt{a^2+b^2}},\ \sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$$

$$-\sqrt{a^2+b^2} \le a\sin x + b\cos x \le \sqrt{a^2+b^2}$$

Practice set

Easy to hard. For each, find $\sqrt{a^2+b^2}$ first. The answers start on the next page.

Q1. Quiz from the Short. What is the maximum of $8\sin x - 6\cos x$? (A) 14 (B) 2 (C) 10.

Q2. Find the greatest and the least value of $5\sin x + 12\cos x$.

Q3. Find the maximum of $\sin x + \cos x$.

Q4. Find the range of $3\sin x + 4\cos x + 2$.

Q5. For which values of k does the equation $3\sin x + 4\cos x = k$ have a solution?

Q6. Find the maximum of $3\cos x + 4\sin\left(x + \frac{\pi}{6}\right)$.

Watch the animated lesson on YouTube

Maximum of a sin x + b cos x (Short)

Scan the code to open the JEEVisionX channel on YouTube. This lesson is in the JEE Concepts Visualization Shorts playlist.

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Answers

Q1

(C) 10

$\sqrt{8^2+(-6)^2}=\sqrt{64+36}=\sqrt{100}=10$. The minus sign does not matter: it is squared. (A) $14=8+6$ is the trap: the two terms are never largest together.

Q2

Greatest 13**, least** -13

$\sqrt{5^2+12^2}=\sqrt{169}=13$.

Q3

$\sqrt{2}$

$\sqrt{1^2+1^2}=\sqrt{2}$, reached at $x=\dfrac{\pi}{4}$, where $\sin x=\cos x=\dfrac{1}{\sqrt{2}}$. Not $1+1=2$.

Q4

From -3 **to** 7

$3\sin x+4\cos x$ runs from -5 to 5; adding 2 gives -3 to 7.

Q5

$-5\le k\le 5$

The left side takes every value from -5 to 5 and no other, so there is a solution exactly when k is in that range.

Q6

$\sqrt{37}$

$4\sin\left(x+\frac{\pi}{6}\right)=2\sqrt{3}\sin x+2\cos x$, so the expression is $2\sqrt{3}\sin x+5\cos x$, with maximum $\sqrt{12+25}=\sqrt{37}$.

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