Why Is $\sin^{-1}(\sin x)$ Not Always x?

Mathematics | JEE Main & Advanced

Don't Memorize. Visualize.

Take $x = 2$. Then $\sin 2 \approx 0.909$, but $\sin^{-1}(0.909) \approx 1.14$, not 2. Many angles have the same sine, and $\sin^{-1}$ gives only one of them: the angle from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. So $\sin^{-1}(\sin x) = x$ only when x is already in that range. For any other x, find the angle in that range with the same sine.

RADIANS

Here x is an angle in radians. A half turn is π, about 3.14, so 2 radians is about $115°$. $\frac{\pi}{2}$ is a quarter turn, about 1.57.

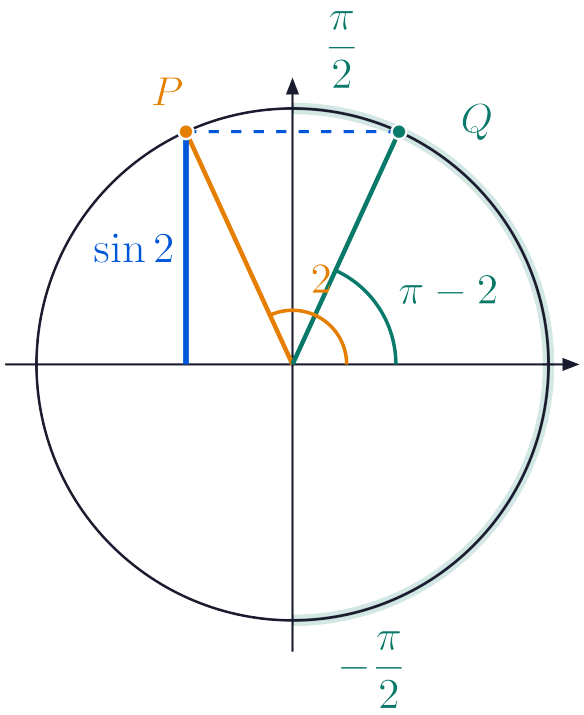
PRINCIPAL VALUE

$\sin^{-1} y$ is the one angle θ with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $\sin\theta = y$.

It is defined for $-1 \le y \le 1$.

Many angles, one sine

On a circle of radius 1, the point P at angle 2 has height $\sin 2 \approx 0.909$.
Mirror P in the vertical axis. It lands on Q, at angle $\pi - 2$, at the same height. So $\sin(\pi - 2) = \sin 2$. One full turn more also ends at P: $\sin(2 + 2\pi) = \sin 2$. Many angles have this same sine.

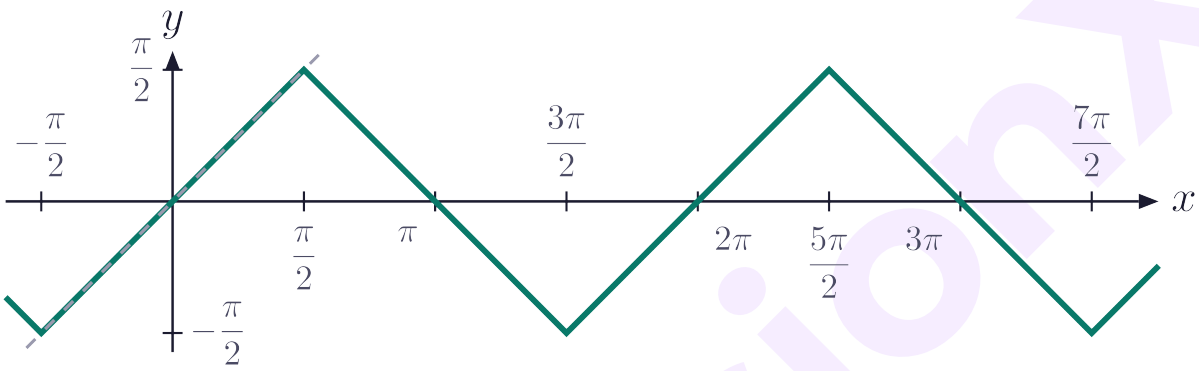


$\sin^{-1}$ answers on the right half

On the right half of the circle, from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, every height from -1 to 1 comes exactly once. Q is there, so $\sin^{-1}(\sin 2) = \pi - 2 \approx 1.14$.

The zigzag graph

Now let x move. From $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ the answer is x itself. From $\frac{\pi}{2}$ to $\frac{3\pi}{2}$ the point P is on the left half, the answer is $\pi - x$, and it falls. Then it rises again as $x - 2\pi$, and so on. The graph of $y = \sin^{-1}(\sin x)$ is a zigzag. It never goes above $\frac{\pi}{2}$ or below $-\frac{\pi}{2}$, and it meets the dashed line $y = x$ only from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.



Reading the zigzag

Every piece crosses the x-axis at a multiple of π and has slope 1 or -1. The corners are at odd multiples of $\frac{\pi}{2}$.

x from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$: $y = x$

x from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$: $y = \pi - x$

x from $\frac{3\pi}{2}$ to $\frac{5\pi}{2}$: $y = x - 2\pi$

x from $\frac{5\pi}{2}$ to $\frac{7\pi}{2}$: $y = 3\pi - x$

From the video

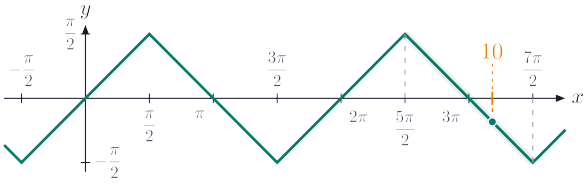
Find $\sin^{-1}(\sin 10)$.

Where is 10**?** $\frac{5\pi}{2} \approx 7.85$ and $\frac{7\pi}{2} \approx 11.0$, so 10 lies between them.

The piece. It crosses the x-axis at 3π and falls: $y = 3\pi - x$.

Check. $3\pi - 10 \approx -0.58$, which lies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

$\sin^{-1}(\sin 10) = 3\pi - 10 \approx -0.58$



CAREFUL

$\sin^{-1}(\sin 10)$ **is not** 10**.** The answer of $\sin^{-1}$ always lies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, and 10 does not. Cancelling $\sin^{-1}$ with $\sin$ works only when x is already in that range.

Find the piece of the zigzag that holds x, then use its line.

Key formulas

Range of $\sin^{-1}$: from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

$\sin^{-1}(\sin x) = x$ only for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

Corners: at odd multiples of $\frac{\pi}{2}$

Same sine: $\sin(\pi - x) = \sin x$, $\sin(x + 2\pi) = \sin x$

Rising piece through $n\pi$, n even: $y = x - n\pi$

Falling piece through $n\pi$, n odd: $y = n\pi - x$

Practice set

Easy to hard. For each, find the piece of the zigzag that holds x and use its line. Question 1 is the quiz at the end of the video. Take $\pi \approx 3.1416$. The answers start on the next page.

Q1. Quiz from the video. What is $\sin^{-1}(\sin 4)$? (A) 4 (B) $\pi - 4$ (C) $4 - \pi$

Q2. Find $\sin^{-1}\left(\sin \frac{2\pi}{3}\right)$.

Q3. Find $\sin^{-1}(\sin 3)$.

Q4. Find $\sin^{-1}(\sin 6)$.

Q5. Find $\sin^{-1}(\sin(-2))$.

Q6. Find $\sin^{-1}(\sin 12)$.

Q7. For which x from 0 to 2π is $\sin^{-1}(\sin x) = \pi - x$?

Q8. For which real x is $\sin^{-1}(\sin x) = x$?

Watch the animated lesson on YouTube

Why Is sin inverse sin x Not Always x?

Scan the code to open the JEEVisionX channel on YouTube. This lesson is in the Maths Concepts Explained Visually playlist.

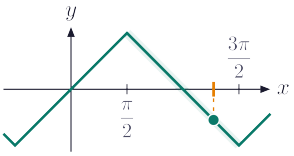
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Answers

Q1

(B) $\pi - 4$

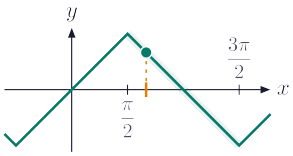
4 lies from $\frac{\pi}{2} \approx 1.57$ to $\frac{3\pi}{2} \approx 4.71$: the piece $y = \pi - x$. So the answer is $\pi - 4 \approx -0.86$. (A): 4 is not from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. (C): $\sin(4 - \pi) = -\sin 4$, the wrong sign.



Q2

$\frac{\pi}{3}$

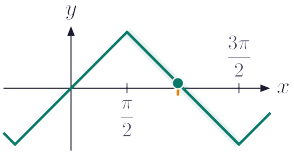
$\frac{2\pi}{3}$ is more than $\frac{\pi}{2}$, so it is not the answer. It lies on the piece $y = \pi - x$: $\pi - \frac{2\pi}{3} = \frac{\pi}{3}$.



Q3

$\pi - 3 \approx 0.14$

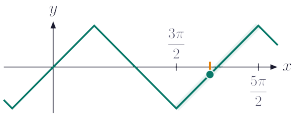
3 lies from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$, on the piece $y = \pi - x$.



Q4

$6 - 2\pi \approx -0.28$

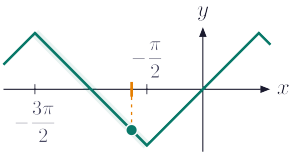
6 lies from $\frac{3\pi}{2} \approx 4.71$ to $\frac{5\pi}{2} \approx 7.85$. That piece crosses the x-axis at 2π and rises: $y = x - 2\pi$.



Q5

$2 - \pi \approx -1.14$

-2 lies from $-\frac{3\pi}{2}$ to $-\frac{\pi}{2}$. That piece crosses at $-\pi$ and falls: $y = -\pi - x$, and $-\pi - (-2) = 2 - \pi$.



Q6

$12 - 4\pi \approx -0.57$

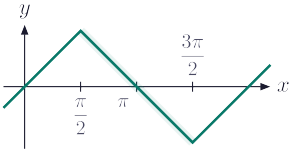
$\frac{7\pi}{2} \approx 11.00$ and $\frac{9\pi}{2} \approx 14.14$. That piece crosses at 4π and rises: $y = x - 4\pi$.



Q7

$\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$

That is the falling piece through π. On the rest of 0 to 2π the zigzag follows another line.



Q8

$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

Only there does the zigzag lie on the line $y = x$. Outside it, x is beyond the range of $\sin^{-1}$.

