# What Is the Difference Between $|f(x)|$ and $f(|x|)$?

Mathematics | JEE Main & Advanced

**Don't Memorize. Visualize.**

Take $f(x) = x^2 - 2x - 3$ and $x = -2$. Then $|f(-2)| = |5| = 5$, but $f(|-2|) = f(2) = -3$. The same two steps in the other order give another answer. The modulus outside acts on the height: the part of the graph below the $x$-axis turns up. The modulus inside acts on $x$: the left part is thrown away and the right part is copied to the left.

**OUTSIDE: $|f(x)|$**

First find $f(x)$, then take its modulus. A height of $5$ stays $5$, and a height of $-3$ becomes $3$.

So $|f(x)|$ is never below the $x$-axis.

**INSIDE: $f(|x|)$**

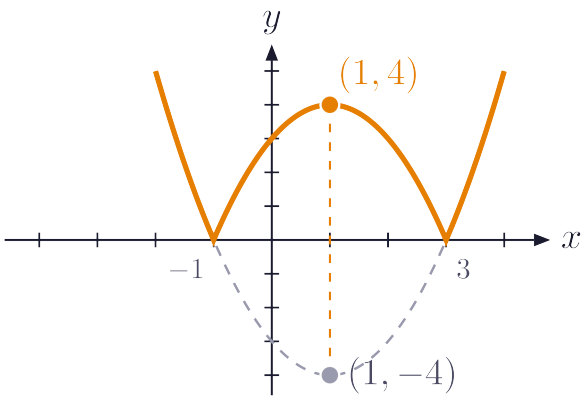
First take $|x|$, then put it into $f$. At $x = -2$ and at $x = 2$ you put in the same number, $2$.

So $f(|x|)$ has the same height at $x$ and $-x$.

## The modulus outside: turn up what is below

The graph of $f(x) = x^2 - 2x - 3 = (x + 1)(x - 3)$ (dashed) crosses the $x$-axis at $-1$ and $3$, and it is below the axis between them. Its lowest point is $(1, -4)$.

$|f(x)|$ keeps every height that is $0$ or more and turns every negative height into the same positive height. So the dip between $-1$ and $3$ turns over the $x$-axis like a page, and the lowest point $(1, -4)$ goes up to $(1, 4)$.

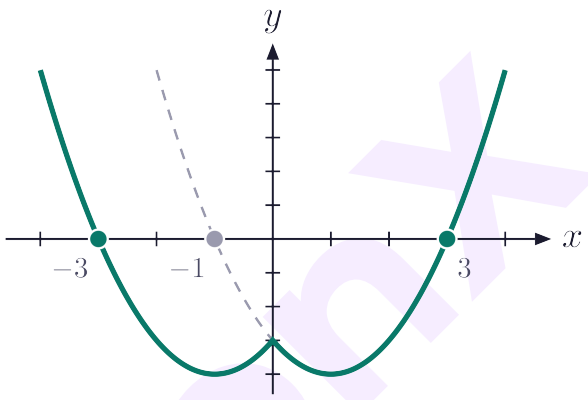



**To draw $y = |f(x)|$**

Keep the part of $y = f(x)$ on or above the $x$-axis. Turn the part below the $x$-axis up, over the $x$-axis.

## The modulus inside: keep the right part, copy it to the left

For $x \geq 0$, $|x| = x$, so $f(|x|) = f(x)$: the right part of the graph stays.
For $x < 0$, $f(|x|) = f(-x)$: at $-2$ the height is $f(2) = -3$, the height at $+2$. So every point on the left copies its height from the matching point on the right. The old left part of $f$ is thrown away, and the right part turns over the $y$-axis.
$f(-1) = 0$, but $f(|-1|) = f(1) = -4$: the root $-1$ is lost. The roots of $f(|x|)$ are $-3$ and $3$.




**To draw $y = f(|x|)$**

Keep the part of $y = f(x)$ for $x \geq 0$. Throw away the part for $x < 0$, and copy the right part over the $y$-axis.

**From the video**

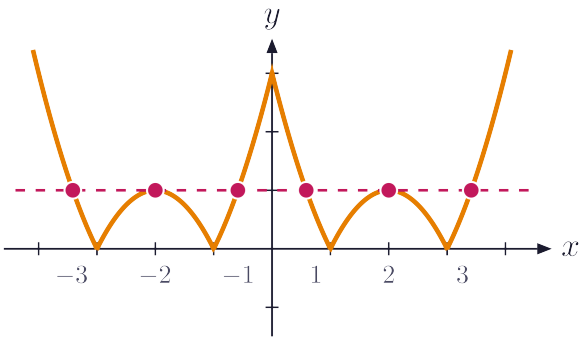
Find the number of real solutions of $\left|x^2 - 4|x| + 3\right| = 1$.

**Spot $f(|x|)$.** $x^2 = |x|^2$, so $x^2 - 4|x| + 3 = g(|x|)$ with $g(x) = x^2 - 4x + 3 = (x-1)(x-3)$.

**Both moves.** Draw $g$, copy its right part to the left for $g(|x|)$, then turn up the parts below the $x$-axis for $\left|g(|x|)\right|$.

**Count.** The line $y = 1$ meets the graph at six points. The two humps reach exactly $1$.

**Check.** $g(|x|) = 1$ gives $|x| = 2 \pm \sqrt{2}$: four values of $x$. $g(|x|) = -1$ gives $|x| = 2$: two values.




**6 real solutions**

**CAREFUL**

**Do not keep the old left part of $f(x)$ in $f(|x|)$.** The graph of $f(|x|)$ is always the same on both sides of the $y$-axis; the left part of $f$ plays no part in it. And $|f(x)|$ moves no point that is on or above the $x$-axis: it only turns up what is below.
**Outside: turn up the part below the $x$-axis. Inside: keep the right part and copy it over the $y$-axis.**

**Key formulas**

Outside: $|f(x)| = f(x)$ where $f(x) \geq 0$, and $-f(x)$ where $f(x) < 0$

Inside: $f(|x|) = f(x)$ for $x \geq 0$, and $f(-x)$ for $x < 0$

$|f(x)|$ is never below the $x$-axis

$f(|x|)$ is the same on both sides of the $y$-axis

$x^2 = |x|^2$, so $x^2 + b|x| + c$ is $f(|x|)$ with $f(x) = x^2 + bx + c$

Number of solutions of $h(x) = k$: the points where $y = k$ meets $y = h(x)$

## Practice set

Easy to hard. Draw the graph first, with each modulus in its own step. In Questions 2 to 6, $f(x) = x^2 - 2x - 3$. Question 1 is the quiz at the end of the video. The answers start on the next page.

**Q1. Quiz from the video.** How many real roots does $x^2 - 5|x| + 6 = 0$ have? (A) $2$ (B) $3$ (C) $4$

**Q2.** Find $|f(2)|$ and $f(|-3|)$.

**Q3.** For which $x$ is $|f(x)|$ different from $f(x)$?

**Q4.** For which $x$ is $f(|x|)$ different from $f(x)$?

**Q5.** How many solutions does $|x^2 - 2x - 3| = 2$ have?

**Q6.** How many solutions does $|x^2 - 2x - 3| = 4$ have?

**Q7.** How many real roots does $x^2 - 2|x| - 3 = 0$ have?

**Q8.** How many real solutions does $\left|x^2 - 4|x| + 3\right| = \frac{1}{2}$ have?

**Watch the animated lesson on YouTube**

What Is the Difference Between |f(x)| and f(|x|)?

Scan the code to open the JEEVisionX channel on YouTube. This lesson is in the Maths Concepts Explained Visually playlist.

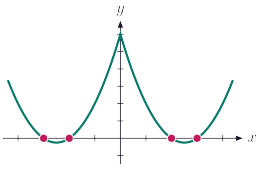
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## Answers

### Q1

**(C)** $4$

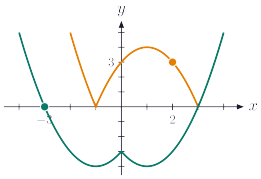
$x^2 = |x|^2$, so $|x|^2 - 5|x| + 6 = 0$: $|x| = 2$ or $3$, and $x = \pm 2, \pm 3$. (A) counts only the roots of $x^2 - 5x + 6 = 0$.



### Q2

$3$ **and** $0$

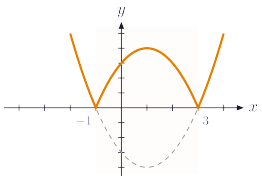
$f(2) = 4 - 4 - 3 = -3$, so $|f(2)| = 3$. $|-3| = 3$ and $f(3) = 9 - 6 - 3 = 0$.



### Q3

$-1 < x < 3$

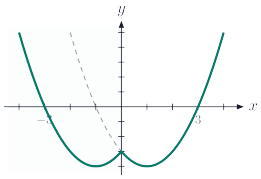
They differ only where $f(x)$ is negative: between the roots $-1$ and $3$. Elsewhere the modulus changes nothing.



### Q4

$x < 0$

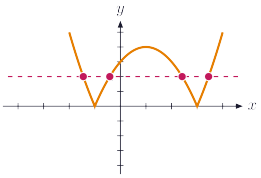
For $x < 0$, $f(|x|) - f(x) = f(-x) - f(x) = 4x$, which is not $0$. For $x \geq 0$ they are equal.



### Q5

$4$

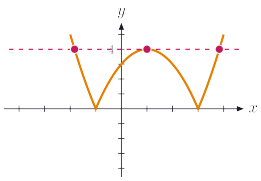
$f(x) = 2$: $x = 1 \pm \sqrt{6}$.
$f(x) = -2$: $x = 1 \pm \sqrt{2}$.
The hump reaches $4$, so the line $y = 2$ cuts it twice.



### Q6

$3$

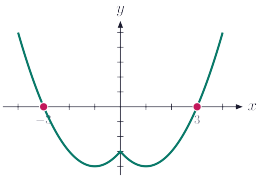
$f(x) = 4$: $x = 1 \pm 2\sqrt{2}$.
$f(x) = -4$: only $x = 1$, where the top of the hump just touches the line $y = 4$.



### Q7

$2$

It is $f(|x|) = 0$: $|x| = 3$ or $|x| = -1$. Only $|x| = 3$ is possible, so $x = \pm 3$. The root $-1$ of $f$ is lost.



### Q8

$8$

$g(|x|) = \frac{1}{2}$: $|x| = 2 \pm \sqrt{\frac{3}{2}}$.
$g(|x|) = -\frac{1}{2}$:
$|x| = 2 \pm \sqrt{\frac{1}{2}}$. All four are positive: eight values of $x$.

