Why Is the Integral of $f(x)$ Equal to the Integral of $f(a+b-x)$?

Mathematics | JEE Main & Advanced

Don't Memorize. Visualize.

The graphs of $f(x)$ and $f(a+b-x)$ are different, but from a to b their areas are equal. The reason: x and $a+b-x$ are always the same distance from the middle, $\frac{a+b}{2}$. So the graph of $f(a+b-x)$ is the mirror image of the graph of f in the line $x=\frac{a+b}{2}$. The region turns over that line like a page, and its area does not change. This result is often called the King's property.

DEFINITE INTEGRAL

For $f(x) \geq 0$, the integral of $f(x)$ from a to b is the area under the graph of f from $x=a$ to $x=b$.

The proof below works for every continuous f.

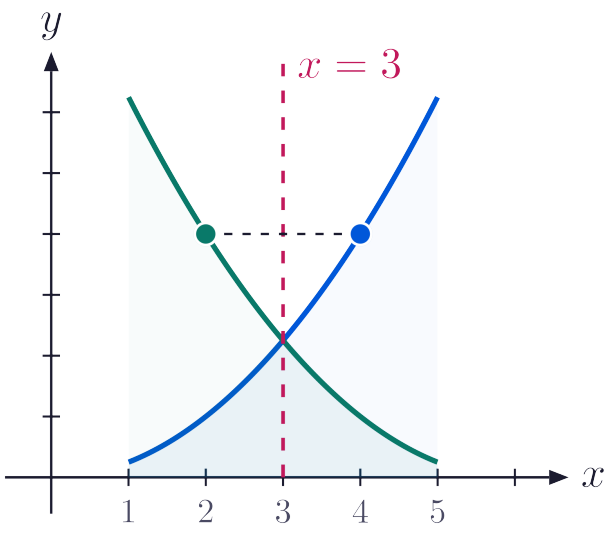
THE MIDDLE LINE

$x=\frac{a+b}{2}$: the line halfway between a and b.

The average of x and $a+b-x$ is always $\frac{a+b}{2}$.

The mirror picture

Take $f(x)=\frac{x^2}{4}$ with $a=1$ and $b=5$. Here $a+b-x=6-x$, so the new function is $f(6-x)$. At $x=2$, $6-x=4$, so the new function is $f(4)=4$. The new graph takes the height of f at 4 and carries it across to 2. Both 2 and 4 are 1 unit from 3, the middle of 1 and 5.



The graph of $f(a+b-x)$ is a mirror image

It is the mirror image of the graph of f in the line $x=\frac{a+b}{2}$. The ends swap: the height at a moves to b, and the height at b moves to a. A region that is turned over keeps its area, so both integrals are $\frac{31}{3}$ here.

The same result by algebra

Substitution $t = a + b - x$

Then $dt = -dx$. When $x = a$, $t = b$; when $x = b$, $t = a$. So

$$\int_a^b f(a+b-x)\,dx = \int_b^a f(t)\,(-dt) = \int_a^b f(t)\,dt = \int_a^b f(x)\,dx.$$

The minus sign swaps the limits back, and the letter inside a definite integral does not change its value. This holds for every f that is continuous on $[a, b]$.

From the video

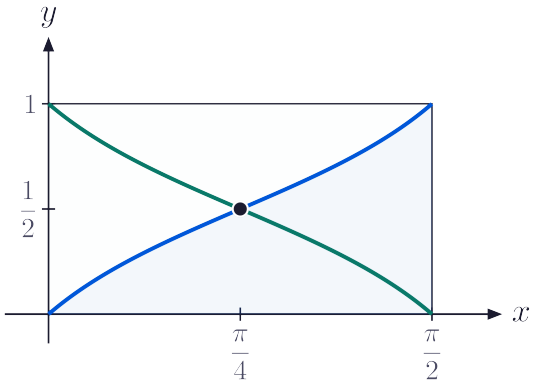
Evaluate $I = \displaystyle\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx$.

Mirror it. Here $a + b = \dfrac{\pi}{2}$. Replace x by $\dfrac{\pi}{2} - x$: $\sin x$ becomes $\cos x$ and $\cos x$ becomes $\sin x$. So, as well,

$$I = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\,dx.$$

Add the two forms. At every x the two heights add up to $\dfrac{\sin x + \cos x}{\sin x + \cos x} = 1$. So $2I$ is the area of a rectangle of height 1 and width $\dfrac{\pi}{2}$: $2I = \dfrac{\pi}{2}$.

$I = \dfrac{\pi}{4}$

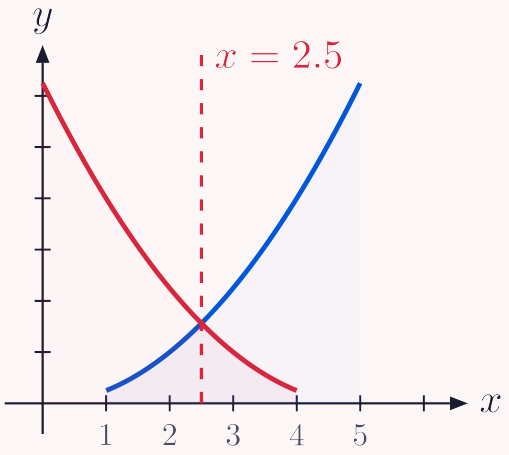


CAREFUL

Replace x by $a + b - x$, not by $b - x$. For the limits 1 and 5, the function $f(5 - x)$ turns the region over $x = 2.5$, not over the middle line $x = 3$. The region moves to 0 to 4, off the limits, and the area changes: $\dfrac{16}{3}$ instead of $\dfrac{31}{3}$.

$b - x$ is right only when $a = 0$.

Lower limit plus upper limit, minus x.



Key formulas

King's property:
$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

When $a = 0$: $\int_0^c f(x)\,dx = \int_0^c f(c-x)\,dx$

Add the two forms:
$$2I = \int_a^b \big[f(x) + f(a+b-x)\big]\,dx$$

If $f(x) + f(a+b-x) = 1$: $I = \dfrac{b-a}{2}$

Practice set

Easy to hard. In each, replace x by $a+b-x$, then add the two forms of I. Question 1 is the quiz at the end of the video. The answers start on the next page.

Q1. Quiz from the video. What is $\displaystyle\int_2^4 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{6-x}}\,dx$? (A) 2 (B) 1 (C) 3

Q2. Evaluate $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\,dx$.

Q3. Evaluate $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$.

Q4. Evaluate $\displaystyle\int_1^3 \frac{x^2}{x^2+(4-x)^2}\,dx$.

Q5. Evaluate $\displaystyle\int_0^{\pi} x\sin x\,dx$ using the property.

Q6. Evaluate $\displaystyle\int_0^{\pi/2} \sin^2 x\,dx$ using the property.

Q7. With $f(x) = \dfrac{x^2}{4}$, which one is equal to $\displaystyle\int_1^5 f(x)\,dx$?

(A) $\displaystyle\int_1^5 f(5-x)\,dx$ (B) $\displaystyle\int_1^5 f(6-x)\,dx$ (C) $\displaystyle\int_1^5 f(4-x)\,dx$

Q8. Evaluate $\displaystyle\int_0^{\pi/4} \ln(1+\tan x)\,dx$.

Watch the animated lesson on YouTube
Why Is the Integral of f(x) Equal to the Integral of f(a+b-x)?
Scan the code to open the JEEVisionX channel on YouTube. This lesson is in the Maths Concepts Explained Visually playlist.

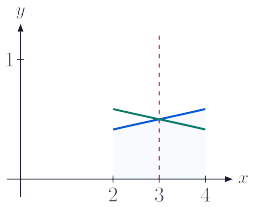
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Answers

Q1

(B) 1

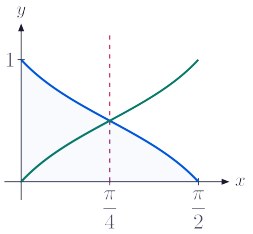
$a+b=6$: the mirror is $\dfrac{\sqrt{6-x}}{\sqrt{6-x}+\sqrt{x}}$, and the two add to 1. So $2I = 4-2 = 2$ and $I = 1$. (A) is $2I$, not halved.



Q2

$\dfrac{\pi}{4}$

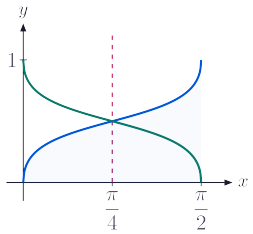
It is the mirror form of the example, so it has the same value: $2I = \dfrac{\pi}{2}$.



Q3

$\dfrac{\pi}{4}$

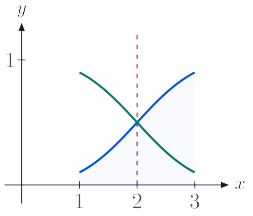
Replace x by $\dfrac{\pi}{2}-x$: $\sin$ and $\cos$ swap. The two forms add to 1, so $2I = \dfrac{\pi}{2}$.



Q4

1

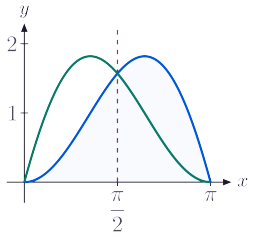
$a+b=4$: the mirror is $\dfrac{(4-x)^2}{(4-x)^2+x^2}$. The two add to 1, so $2I = 3-1 = 2$.



Q5

π

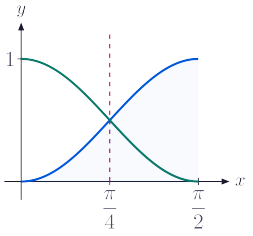
The mirror is $(\pi-x)\sin(\pi-x) = (\pi-x)\sin x$. Adding, $2I = \pi\displaystyle\int_0^{\pi}\sin x\,dx = 2\pi$.



Q6

$\dfrac{\pi}{4}$

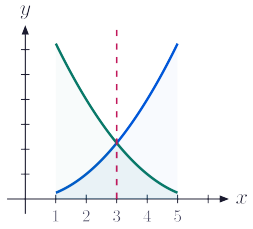
The mirror of $\sin^2 x$ is $\cos^2 x$, and $\sin^2 x+\cos^2 x = 1$. So $2I = \dfrac{\pi}{2}$.



Q7

(B)

$a+b = 1+5 = 6$. (A) gives $\dfrac{16}{3}$ and (C) gives $\dfrac{7}{3}$; only (B) gives $\dfrac{31}{3}$.



Q8

$\dfrac{\pi}{8}\ln 2 \approx 0.272$

$1+\tan\left(\dfrac{\pi}{4}-x\right) = \dfrac{2}{1+\tan x}$, so the two forms add to $\ln 2$. $2I = \dfrac{\pi}{4}\ln 2$.

