Locus of Complex Numbers: Ellipse and Hyperbola

Mathematics | JEE Main & Advanced

Don't Memorize. Visualize.

$|z-a|+|z-b|=k$ says that the distances from z to two fixed points a and b add up to k. $\big||z-a|-|z-b|\big|=k$ says that they differ by k. Which curve you get depends only on how k compares with the gap $|a-b|$ between the two points: no conversion to x and y is needed.

DISTANCE AND FOCI

$|z-a|$ is the distance between the points z and a. The two fixed points a and b are called the foci; each one is a focus.

Read each focus with its sign:

$|z+3|=|z-(-3)|$, so the focus is -3.

ELLIPSE

Pin the two ends of a string at the foci, keep it tight with a pencil and take the pencil all the way round. The curve it draws is an ellipse.

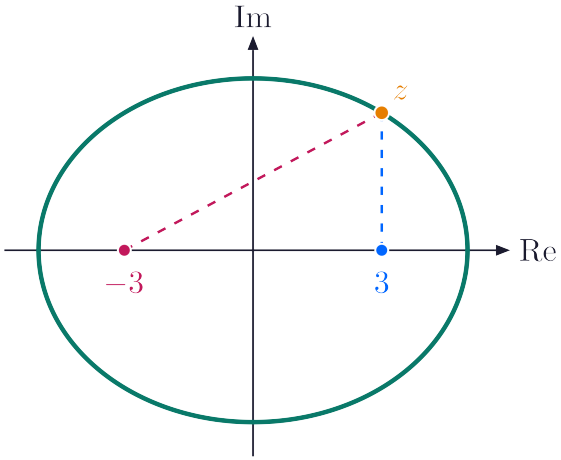
Every point of it has the same sum of distances: the length of the string.

A fixed sum of distances: the ellipse

Take $|z-3|+|z+3|=10$. The foci are 3 and -3, and the distances from z to them add up to 10.
At $z=3+3.2i$ they are 3.2 and 6.8; at $z=4i$ both are 5; at $z=5$ they are 2 and 8. Each time the sum is 10.

The locus is an ellipse

The distances are the two pieces of a string of length 10 pinned at 3 and -3. As z goes round, it draws an ellipse with foci 3 and -3.

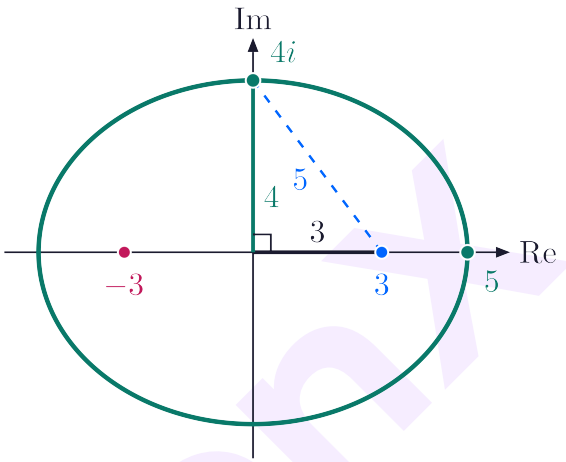


How wide and how tall?

Width. On the real axis to the right of 3, $z = x$ and the distances are $x - 3$ and $x + 3$. They add up to $2x = 10$, so $x = 5$: half the length of the string. By symmetry the ellipse also meets the real axis at -5.

Height. At the top point both distances are equal, so each is 5. The right triangle with base 3 (half the gap) and slanted side 5 has height $\sqrt{5^2 - 3^2} = 4$. The ellipse reaches $4i$ and $-4i$.

In conic sections it is $\frac{x^2}{25} + \frac{y^2}{16} = 1$; the distances gave its size without it.



Any ellipse $|z - a| + |z - b| = k$ with $k > |a - b|$

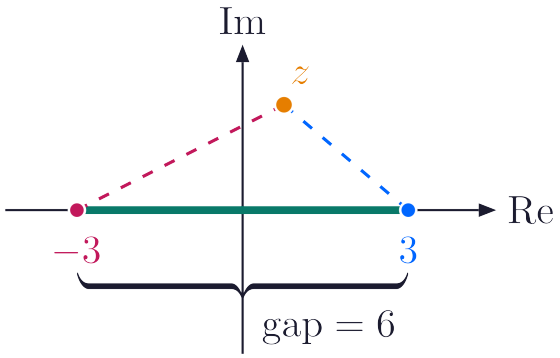
Centre $\frac{a+b}{2}$. Half-width $\frac{k}{2}$ along the line through the foci. Half-height $\frac{1}{2}\sqrt{k^2 - |a-b|^2}$ across it, from the right triangle.

The string as long as the gap: a line segment

Shorten the string. At $k = 6$ it is exactly as long as the gap between 3 and -3, so it lies flat between the foci, and the ellipse becomes the line segment from -3 to 3.

Why. Join any z to 3 and -3. In a triangle two sides together are longer than the third, so off the segment the sum is more than 6. On the segment the triangle is flat and the sum is exactly 6.

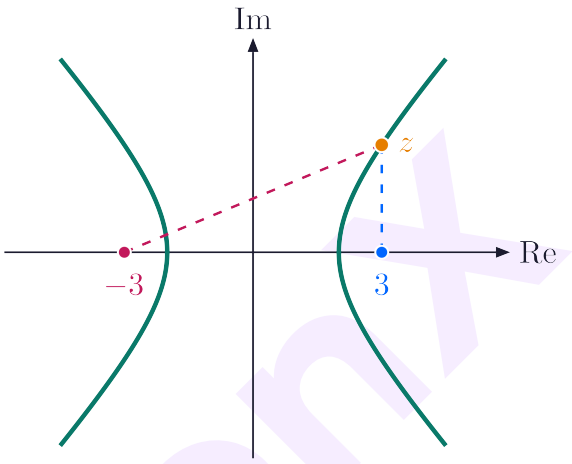
So $|z-3|+|z+3|$ is never less than 6, and $|z-3|+|z+3| = 4$ has no point at all.



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A fixed difference: the hyperbola

Take $\big||z-3|-|z+3|\big|=4$. The outer modulus means: the larger distance minus the smaller one is 4.
At $z=3+2.5i$ the distances are 2.5 and 6.5; at $z=2$ they are 1 and 5. These points make a curve near 3. The difference can also be 4 the other way round, with z nearer -3 (at $z=-2$: 5 and 1), which gives a second curve near -3.

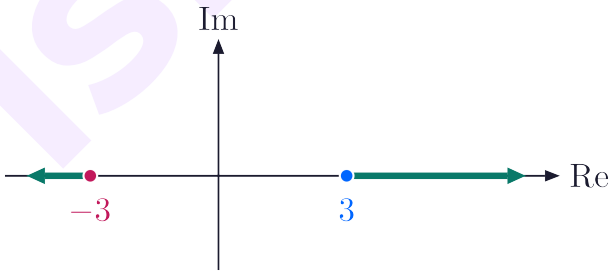


The locus is a hyperbola

The two curves together are a hyperbola with foci 3 and -3: $\dfrac{x^2}{4}-\dfrac{y^2}{5}=1$ in conic sections.

The difference as large as the gap: two rays

In a triangle the difference of two sides is less than the third side, so the difference of the distances is never more than 6. It is exactly 6 only on the real axis where $x\geq 3$ or $x\leq -3$: at $z=4$ the distances are 1 and 7, at $z=6$ they are 3 and 9. So $\big||z-3|-|z+3|\big|=6$ is the two rays $x\geq 3$ and $x\leq -3$ of the real axis, and a difference of 8 gives no point.



CAREFUL

$|z-3|+|z+3|=6$ **is not an ellipse.** The gap between the foci is also 6: the string only reaches from one focus to the other, so the locus is the line segment from -3 to 3. Many students call it an ellipse. And when the sum is less than the gap, as in $|z-3|+|z+3|=4$, there is no point at all.
Before you name the curve, compare k with the gap $|a-b|$.

Compare k with the gap $|a-b|$

	Sum $\lvert z-a\rvert+\lvert z-b\rvert=k$	**Difference** $\big\lvert\lvert z-a\rvert-\lvert z-b\rvert\big\rvert=k$
k more than the gap	an ellipse	no point
k equal to the gap	the segment from a to b	two rays, from a and from b, outward
k less than the gap	no point	a hyperbola, for $k>0$

A difference of 0 means $|z-a|=|z-b|$: the perpendicular bisector of the segment joining the foci.

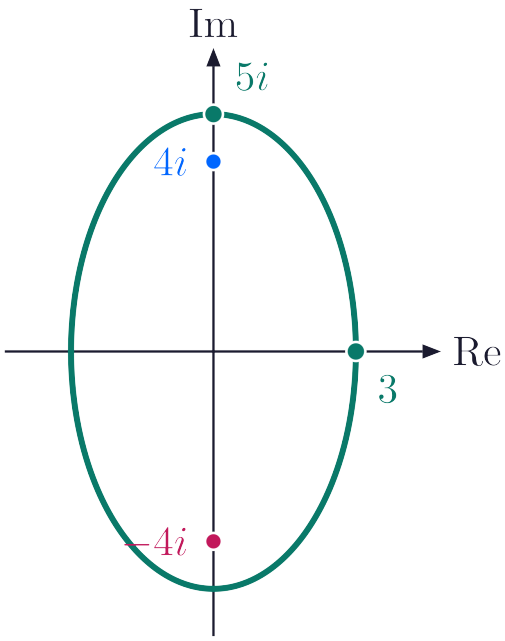
From the video

Find the locus of z if $|z-4i|+|z+4i|=10$.

Compare. The foci are $4i$ and $-4i$; the gap is 8, and the sum 10 is more than 8: an ellipse. It stands upright, because the foci are on the imaginary axis.

Check two points. At $5i$ the distances are 1 and 9. At 3 both are 5 (the 3, 4, 5 right triangle). Each pair adds up to 10.

An ellipse with foci $\pm 4i$, through $\pm 5i$ and ± 3: $\frac{x^2}{9}+\frac{y^2}{25}=1$.



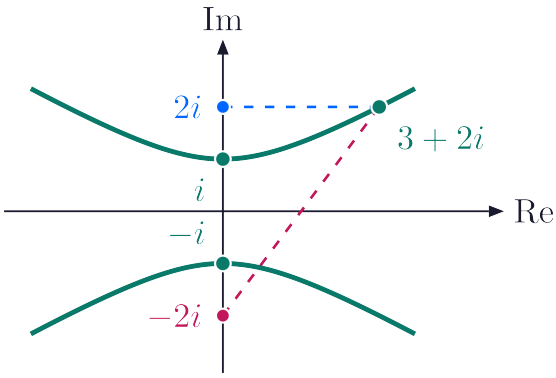
One more, with the difference

Find the locus of z if $\big||z-2i|-|z+2i|\big|=2$.

Compare. The foci are $2i$ and $-2i$; the gap is 4, and the difference 2 is more than 0 and less than 4: a hyperbola. Its two curves open up and down.

Check two points. At i the distances are 1 and 3; at $3+2i$ they are 3 and 5. Each difference is 2.

A hyperbola with foci $\pm 2i$, through $\pm i$: $y^2-\frac{x^2}{3}=1$.



Key formulas

Distance: $|z-a|$, from z to a

Gap between the foci: $|a-b|$

Sum, $k>|a-b|$: an ellipse

Sum, $k=|a-b|$: the segment from a to b

Half-width: $\frac{k}{2}$; half-height: $\frac{1}{2}\sqrt{k^2-|a-b|^2}$

Difference, $0<k<|a-b|$: a hyperbola

Difference, $k=|a-b|$: two rays

Sum less than, or difference more than, the gap: no point

Practice set

Easy to hard. For each, find the foci and the gap first, compare k with the gap, then name the locus. Questions 1, 2, 4 and 5 are the quizzes at the end of the video and its Shorts. The answers start on the next page.

Q1. Quiz from the Short. Which point lies on $|z-4|+|z+4|=10$? (A) $3i$ (B) $4i$ (C) $6i$

Q2. Quiz from the Short. What is the locus of $|z-3|+|z+3|=5$? (A) an ellipse (B) a line segment (C) no point at all

Q3. Find the locus of z if $|z-2|+|z+2|=6$, and how far it reaches along each axis.

Q4. Quiz from the Short. How many parts does the locus of $\big||z-1|-|z+1|\big|=1$ have? (A) one (B) two (C) none

Q5. Quiz from the video. What is the locus of z if $\big||z-i|-|z+i|\big|=2$? (A) a hyperbola (B) two rays (C) no point at all

Q6. Find the locus of z if $\big||z-5|-|z+5|\big|=6$, and where it crosses the real axis.

Q7. For which values of k is the locus of $|z-3i|+|z+3i|=k$ an ellipse?

Q8. A student says that $|z-3|-|z+3|=4$ (no outer modulus) is a whole hyperbola. Find the mistake, and the correct locus.

Watch the animated lesson on YouTube
Locus of Complex Numbers: Ellipse and Hyperbola
Scan the code to open the JEEVisionX channel on YouTube. This lesson is in the Complex Numbers playlist.

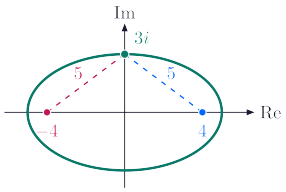
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Answers

Q1

(A) $3i$

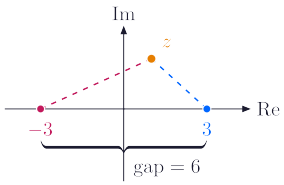
The foci are 4 and -4. From $3i$ each is $\sqrt{4^2+3^2}=5$ away: $5+5=10$. At $4i$ the sum is $2\sqrt{32}\approx 11.3$, at $6i$ it is $2\sqrt{52}\approx 14.4$.



Q2

(C) no point at all

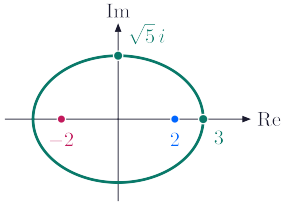
The gap between 3 and -3 is 6. In the triangle z, 3, -3 the sum is at least 6, so it can never be 5.



Q3

An ellipse, $\frac{x^2}{9}+\frac{y^2}{5}=1$

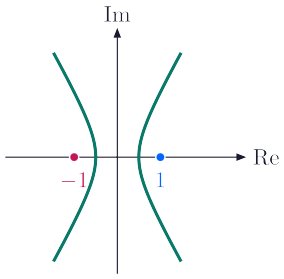
Gap 4, less than 6. Half-width 3, half-height $\frac{1}{2}\sqrt{36-16}=\sqrt{5}$: it reaches ± 3 and $\pm\sqrt{5}\,i$.



Q4

(B) two

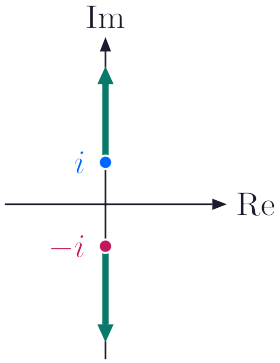
Gap 2, and $0<1<2$: a hyperbola, through $\frac{1}{2}$ and $-\frac{1}{2}$. The imaginary axis, where the difference is 0, lies between its two parts.



Q5

(B) two rays

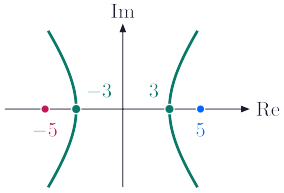
Gap $|i-(-i)|=2$, equal to the difference: the imaginary axis from i upwards and from $-i$ downwards. At $3i$: $4-2=2$.



Q6

A hyperbola, $\frac{x^2}{9}-\frac{y^2}{16}=1$

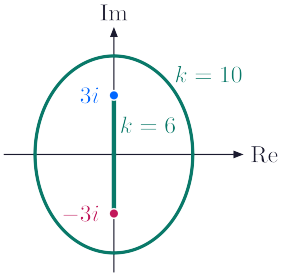
Gap 10, more than 6. Half the difference is 3, so it crosses the real axis at ± 3 (at 3: $8-2=6$); $25-9=16$.



Q7

$k>6$

The gap between $3i$ and $-3i$ is 6. At $k=6$ the locus is the segment between the foci, and for k less than 6 there is no point.



Q8

Only the branch nearer -3

Without the outer modulus, z must be 4 farther from 3 than from -3: the curve through -2 ($5-1=4$). At 2 the difference is $1-5=-4$.

