Greatest and Least Value of $|z|$ on a Circle

Mathematics | JEE Main & Advanced

Don't Memorize. Visualize.

When z lies on the circle $|z-a|=r$, $|z|$ is its distance from the origin. The nearest and the farthest points both lie on the line through the origin and the centre a, so the least value of $|z|$ is $\big||a|-r\big|$ and the greatest is $|a|+r$. The triangle inequality is the same picture, written as algebra.

MODULUS AS A DISTANCE

$|z|$ is the length of the arrow from 0 to z: the distance of z from the origin.

$|z-a|$ is the distance between the points z and a.

THE CIRCLE $|z-a|=r$

Every point at distance r from a: the circle with centre a and radius r.

Read the centre with its sign:

$|z+3-4i|=|z-(-3+4i)|$.

How far can z be from the origin?

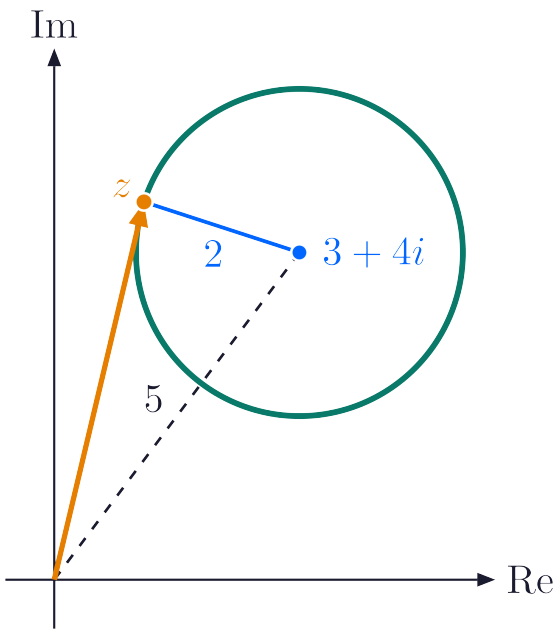
Take the circle $|z-(3+4i)|=2$: centre $a=3+4i$ and radius 2. As z goes round it, its distance $|z|$ from the origin keeps changing.

The origin, the centre and z make a triangle. Its sides are $|a|=|3+4i|=5$, the radius 2, and $|z|$.

A triangle gives the answer

In a triangle, one side is less than the sum of the other two and more than their difference:

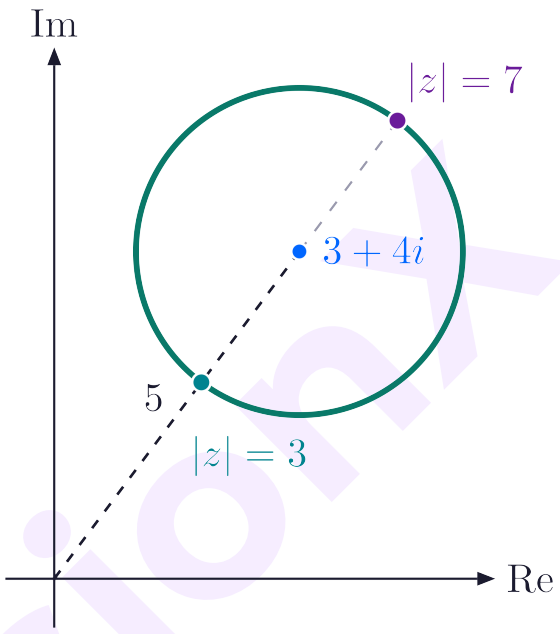
$$5-2\le|z|\le5+2,\quad\text{so}\quad 3\le|z|\le7.$$



The nearest and the farthest points

The two ends of this range are reached when the triangle is flat, that is when z lies on the line through 0 and the centre.
On the far side of the centre, $|z|$ is greatest: $|z| = 7$ at $4.2 + 5.6i$. On the near side it is least: $|z| = 3$ at $1.8 + 2.4i$.
Both points are the centre moved by the radius along that line:

$$(3 + 4i)\left(1 \pm \tfrac{2}{5}\right) = 4.2 + 5.6i \text{ and } 1.8 + 2.4i.$$



The rule, when the origin is outside or on the circle ($|a| \geq r$)

$$|a| - r \;\leq\; |z| \;\leq\; |a| + r$$

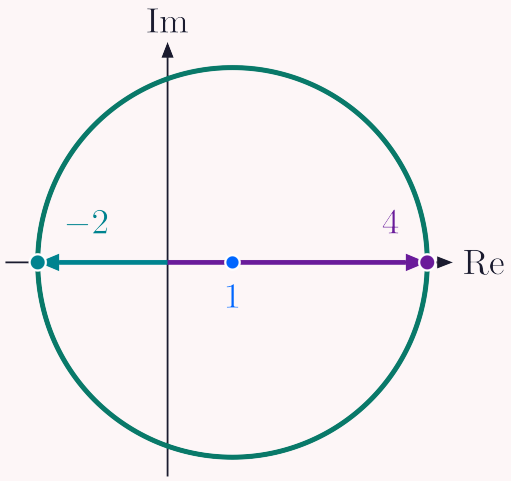
The least value is at $a\left(1 - \dfrac{r}{|a|}\right)$, the greatest at $a\left(1 + \dfrac{r}{|a|}\right)$. When the circle passes through the origin ($|a| = r$), the least value is 0, at the origin itself.

CAREFUL

The origin inside the circle. For $|z - 1| = 3$ the rule gives $|a| - r = 1 - 3 = -2$, and a distance cannot be negative.

Here $|a| = 1$ is less than $r = 3$, so the origin is inside the circle. The nearest point is still on the line through 0 and the centre, but on the other side of the origin: -2, at distance $3 - 1 = 2$. The farthest is 4, at distance $3 + 1 = 4$.

Check first whether $|a| < r$. Then the least value is $r - |a|$.



Both cases at once

$$\text{least } |z| = \big||a| - r\big|, \qquad \text{greatest } |z| = |a| + r.$$

If the centre is the origin ($a = 0$), every point of the circle has $|z| = r$.

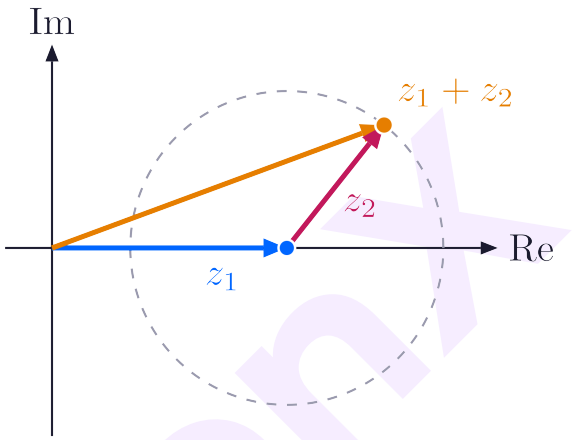
The triangle inequality

Put z_2 at the tip of z_1. Then $z_1 + z_2$ is the third side of a triangle, so for any z_1 and z_2:

The triangle inequality

$$\big||z_1| - |z_2|\big| \le |z_1 + z_2| \le |z_1| + |z_2|$$

The same holds with $z_1 - z_2$ in the middle.



The right side is reached when z_1 and z_2 point the same way, the left side when they point opposite ways. With $z_1 = 3$ and $|z_2| = 2$, $|z_1 + z_2|$ runs from 1 to 5, and the tip of $z_1 + z_2$ runs on the circle $|w - 3| = 2$: the first picture again.

From the video

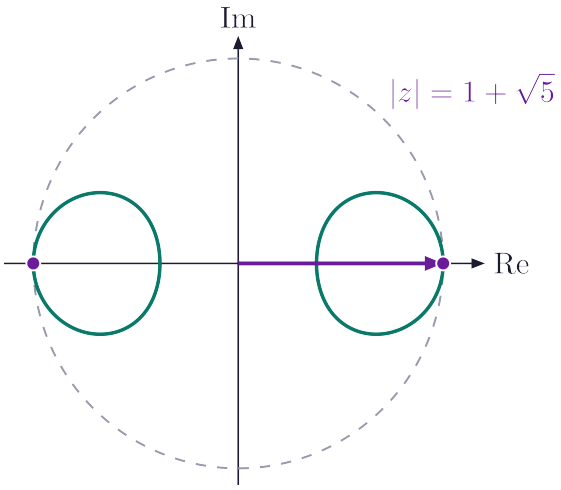
If $\left|z - \dfrac{4}{z}\right| = 2$, find the greatest value of $|z|$.

Triangle inequality. $|z| - \left|\dfrac{4}{z}\right| \le \left|z - \dfrac{4}{z}\right| = 2$.

Solve. With $r = |z| > 0$, $\left|\dfrac{4}{z}\right| = \dfrac{4}{r}$, so $r - \dfrac{4}{r} \le 2$, that is $r^2 - 2r - 4 \le 0$. The roots are $1 \pm \sqrt{5}$, so $r \le 1 + \sqrt{5}$.

Is it reached? At $z = 1 + \sqrt{5}$, $\dfrac{4}{z} = \sqrt{5} - 1$, so $z - \dfrac{4}{z} = 2$. Yes.

The picture: z moves on two closed curves, and the dashed circle $|z| = 1 + \sqrt{5}$ touches them at $\pm(1 + \sqrt{5})$.



The greatest value of $|z|$ is $1 + \sqrt{5}$.

Any such question in three steps

1. Write the circle as $|z - a| = r$ and read the centre a, with its sign.
2. Compare $|a|$ with r: is the origin outside, on, or inside the circle?
3. The least value is $\big||a| - r\big|$ and the greatest $|a| + r$, both on the line through 0 and a. For distances from a point w, use $|a - w|$ in place of $|a|$.

Measuring from a point other than the origin

Distance from a point w

$|z-w|$ is the distance of z from the point w. It is the same picture, measured from w instead of 0. On the circle $|z-a|=r$:

$$\big||a-w|-r\big| \;\le\; |z-w| \;\le\; |a-w|+r.$$

For the disc $|z-a|\le r$ the greatest value is the same, and the least value is 0 when w is in the disc.

A second example

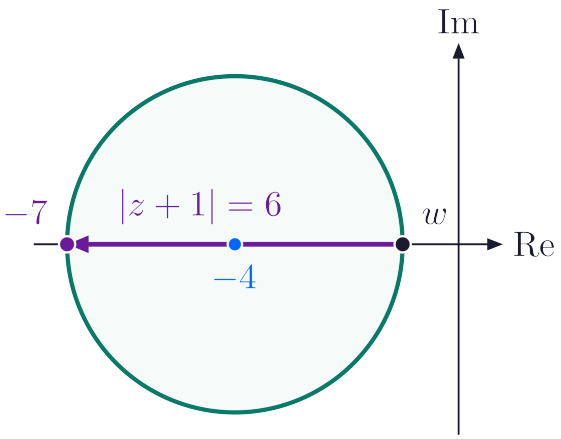
If $|z+4|\le 3$, find the greatest and the least value of $|z+1|$.

Read the picture. $|z+4|\le 3$ is the disc with centre -4 and radius 3. $|z+1|=|z-(-1)|$ is the distance of z from the point $w=-1$.

Greatest. $|a-w|=|-4+1|=3$, so the greatest value is $3+3=6$, at $z=-7$.

Least. $|a-w|=3=r$: the point $w=-1$ is on the circle, so it is in the disc. The least value is 0, at $z=-1$.

The greatest value is 6 and the least is 0.



Key formulas

Distance from 0: $|z|$

Circle: $|z-a|=r$, centre a, radius r

Origin outside or on: $|a|-r\le|z|\le|a|+r$

Origin inside: $r-|a|\le|z|\le r+|a|$

Least: $\big||a|-r\big|$; greatest: $|a|+r$

The points: $a\left(1\pm\dfrac{r}{|a|}\right)$, for $a\neq 0$

Triangle inequality:
$\big||z_1|-|z_2|\big|\le|z_1\pm z_2|\le|z_1|+|z_2|$

From a point w:
$\big||a-w|-r\big|\le|z-w|\le|a-w|+r$

Practice set

Easy to hard. Sketch each question first: mark the centre, decide whether the origin is outside, on or inside the circle, then calculate. The answers start on the next page.

Q1. $|z + 3 - 4i| = 1$. Find the greatest and the least value of $|z|$.

Q2. $|z - 2i| = 5$. Find the greatest and the least value of $|z|$.

Q3. $|z - (1 + i)| = \sqrt{2}$. Find the greatest and the least value of $|z|$.

Q4. $|z - 4 - 5i| = 2$. Find the greatest and the least value of $|z - (1 + i)|$.

Q5. $|z - 3 - 4i| \leq 2$. Find the greatest and the least value of $|z|$.

Q6. $|z_1| = 5$ and $|z_2| = 2$. Find all the values that $|z_1 + z_2|$ can take.

Q7. If $\left|z - \frac{4}{z}\right| = 2$, find the least value of $|z|$.

Q8. A student says that the least value of $|z|$ on $|z - 2| = 5$ is $2 - 5 = -3$. Find the mistake, and the correct least value.

Watch the animated lesson on YouTube

Greatest and Least Value of Mod z on a Circle

Scan the code to open the JEEVisionX channel on YouTube. This lesson is in the Complex Numbers playlist.

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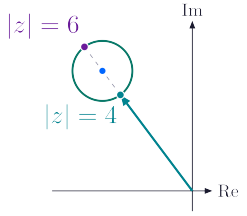
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Answers

Q1

Greatest 6, least 4

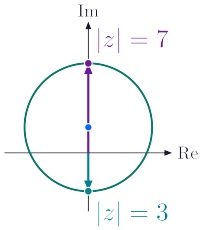
The centre is $-3+4i$, so $|a|=5$ and $r=1$: $5+1$ and $5-1$, at $-3.6+4.8i$ and $-2.4+3.2i$.



Q2

Greatest 7, least 3

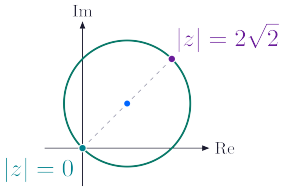
$|2i|=2$ is less than 5: the origin is inside. Least $5-2=3$ at $-3i$, greatest $5+2=7$ at $7i$.



Q3

Greatest $2\sqrt{2}$, least 0

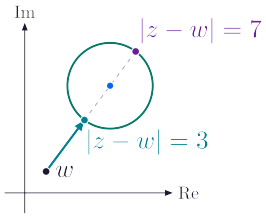
$|1+i|=\sqrt{2}$ is the radius: the circle passes through the origin. Least 0 at 0, greatest $2\sqrt{2}$ at $2+2i$.



Q4

Greatest 7, least 3

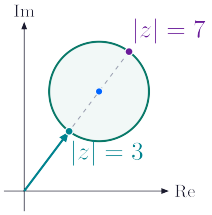
Measure from $w=1+i$: $|(4+5i)-(1+i)|=|3+4i|=5$, so 5 ± 2, at $5.2+6.6i$ and $2.8+3.4i$.



Q5

Greatest 7, least 3

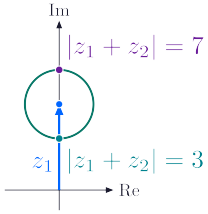
The disc's nearest and farthest points are on its circle, so the answer is the same as for $|z-3-4i|=2$.



Q6

$3\le|z_1+z_2|\le7$

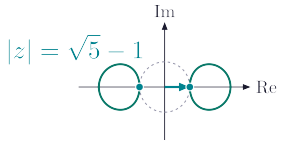
$5-2\le|z_1+z_2|\le5+2$. As z_2 turns, the tip of z_1+z_2 runs on a circle of radius 2 about z_1 (here $z_1=5i$).



Q7

Least $\sqrt{5}-1$

$\left|\frac{4}{z}\right|-|z|\le2$ gives $r^2+2r-4\ge0$, so $r\ge\sqrt{5}-1$. At $z=\sqrt{5}-1$, $\frac{4}{z}=\sqrt{5}+1$ and $z-\frac{4}{z}=-2$.



Q8

Least 3, not -3

$|a|=2$ is less than $r=5$: the origin is inside, so $|a|-r$ is negative. Least $5-2=3$ at -3, greatest 7 at 7.

