Locus of Complex Numbers: Circles and Lines

Mathematics | JEE Main & Advanced

Don't Memorize. Visualize.

$|z-a|$ is the distance between z and the point a. Read a modulus equation as a sentence about distances and the locus draws itself: one fixed distance is a circle, two equal distances are a straight line, and one distance k times another, with k positive and not 1, is a circle again.

DISTANCE $|z-a|$

$z-a$ is the arrow from the point a to the point z, and $|z-a|$ is its length.

So $|z-a|$ is the distance between z and a.

LOCUS

The locus is the set of all the points that obey a rule.

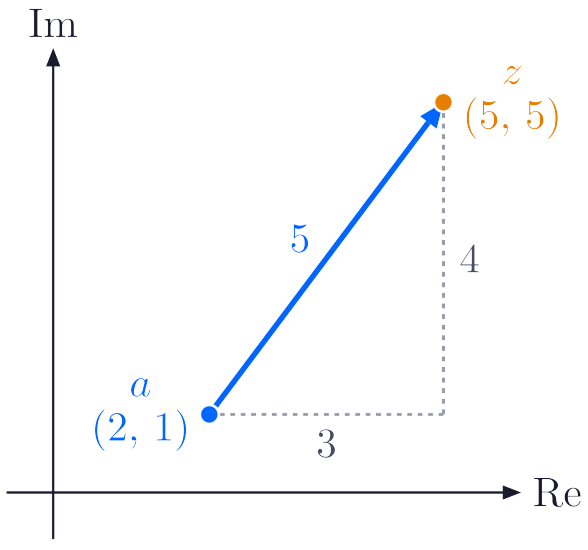
Let a point keep obeying the rule as it moves: the path it traces is the locus.

$|z-a|$ is a distance

Take $a = 2+i$, the point $(2,1)$, and $z = 5+5i$, the point $(5,5)$. Then

$$z-a = 3+4i, \qquad |z-a| = \sqrt{3^2+4^2} = 5.$$

The arrow from a to z goes 3 across and 4 up, so the two points are 5 apart. In general, with $z = x+iy$ and $a = p+iq$, $|z-a| = \sqrt{(x-p)^2+(y-q)^2}$: the distance formula.

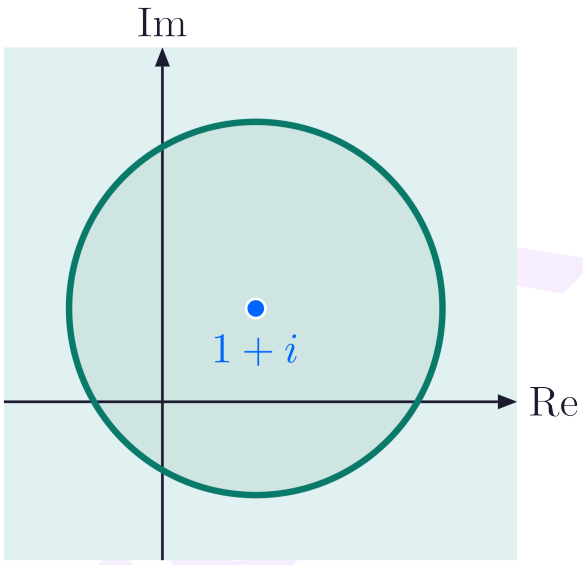


Read the point: write z minus a number

$|z+1| = |z-(-1)|$ is the distance from -1, and $|z+1-i| = |z-(-1+i)|$ is the distance from $-1+i$. A number in front of z comes out first: $|2z+2| = 2\,|z+1|$.

One fixed distance: a circle

$|z-(1+i)|=2$ says that z is always 2 away from $1+i$. Let the point move and keep that distance: it traces the circle with centre $1+i$ and radius 2.
Inside the circle the distance is less than 2: $|z-(1+i)|<2$ (the green region). Outside it is more than 2: $|z-(1+i)|>2$. The origin is inside, since $|0-(1+i)|=\sqrt{2}$, which is less than 2.



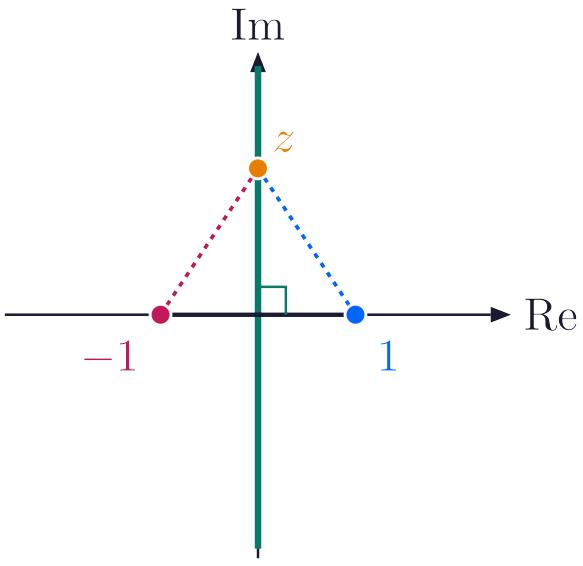
One fixed distance

$|z-a|=r$ $(r>0)$ is the circle with centre a and radius r.

$|z-a|<r$ is the inside, $|z-a|\le r$ includes the circle, and $|z-a|>r$ is the outside.

Two equal distances: a straight line

$|z-1|=|z+1|$. Read $|z+1|$ as $|z-(-1)|$, the distance from -1. So z is as far from 1 as from -1. Let the point move with these two distances equal, and it can only go up and down: the locus is the imaginary axis, the line $x=0$.
It is the perpendicular bisector of the segment from -1 to 1: it passes through the midpoint 0 at a right angle. In x and y: $(x-1)^2+y^2=(x+1)^2+y^2$ gives $-4x=0$, so $x=0$.

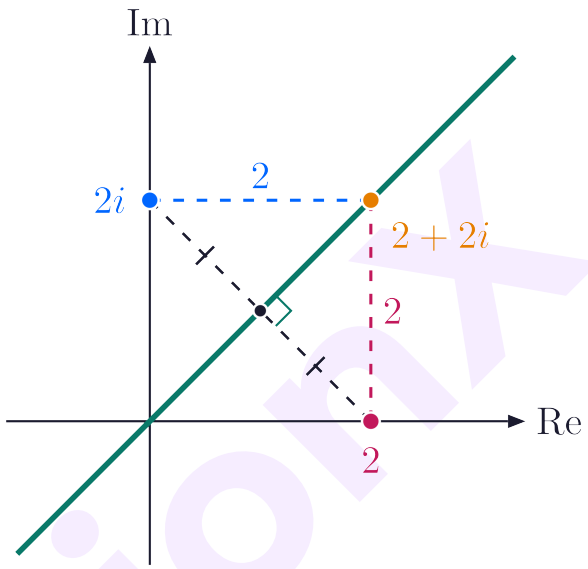


Two equal distances

$|z-a|=|z-b|$ $(a\neq b)$ is the perpendicular bisector of the segment from a to b.

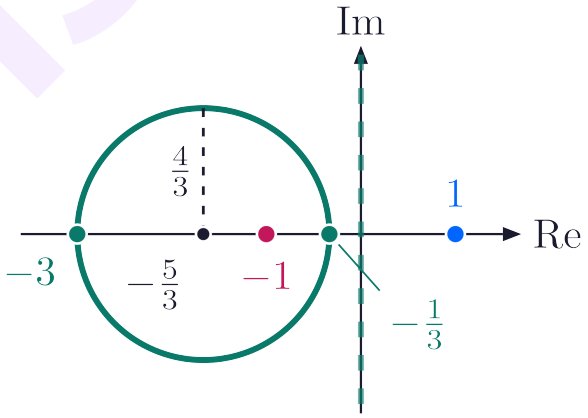
A less obvious line

$|z-2i| = |z-2|$: z is as far from $2i$ as from 2. The locus is the perpendicular bisector of the segment from $2i$ to 2. It passes through the midpoint $1+i$ and is at right angles to the segment: the line $y = x$.
Check with $z = x + iy$: $x^2 + (y-2)^2 = (x-2)^2 + y^2$ gives $4x - 4y = 0$, so $y = x$. And the point $2 + 2i$ is 2 from both: $|2 + 2i - 2i| = 2$ and $|2 + 2i - 2| = 2$.



One distance twice the other: a circle again

Put a 2 in front: $|z-1| = 2|z+1|$. Now z must be twice as far from 1 as from -1. The path is no longer a line: it closes into a circle round -1.
On the real axis it passes through $-\frac{1}{3}$ (distances $\frac{4}{3}$ and $\frac{2}{3}$) and -3 (distances 4 and 2). The centre is halfway between them, $-\frac{5}{3}$, and the radius is half the gap, $\frac{4}{3}$. This is the circle of Apollonius.



Why a circle: the algebra

Put $z = x + iy$ and square both sides:

$$(x-1)^2 + y^2 = 4\left[(x+1)^2 + y^2\right] \implies 3x^2 + 3y^2 + 10x + 3 = 0 \implies \left(x + \tfrac{5}{3}\right)^2 + y^2 = \left(\tfrac{4}{3}\right)^2.$$

The quick way: two points on the line through a and b

For $|z-a| = k|z-b|$ ($k > 0$, $k \neq 1$) the circle crosses the line through a and b at

$$P = \frac{a + kb}{1 + k} \text{ (between } a \text{ and } b\text{)}, \qquad Q = \frac{kb - a}{k - 1} \text{ (outside)}.$$

PQ is a diameter: the centre is halfway between P and Q, and the radius is half of $|P - Q|$. Here $P = -\frac{1}{3}$ and $Q = -3$.

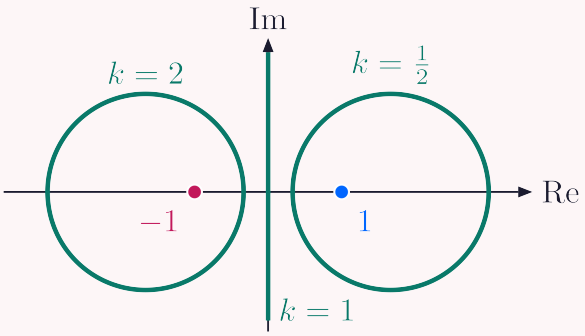
CAREFUL

Read k before you draw. For $|z-1| = k|z+1|$, squaring gives

$$(1-k^2)(x^2+y^2) - 2(1+k^2)x + (1-k^2) = 0.$$

The x^2+y^2 terms carry $1-k^2$, so they disappear only when $k=1$, and then the locus is the line $x=0$. For any other positive k it is a circle.

Look for a hidden k: $|z-1| = |2z+2|$ is $|z-1| = 2|z+1|$, a circle, not a line.



Solved example

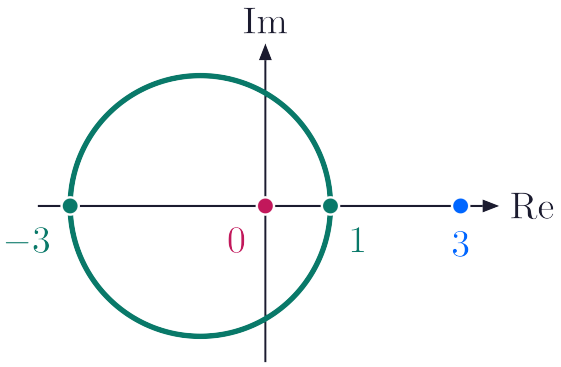
Find the locus of z if $|z-3| = 2|z|$.

Square. With $z = x+iy$: $(x-3)^2+y^2 = 4(x^2+y^2)$, so $3x^2+3y^2+6x-9=0$, that is $x^2+y^2+2x-3=0$.

Complete the square. $(x+1)^2 + y^2 = 2^2$.

Check. The circle meets the real axis at 1 and -3: $|1-3| = 2 = 2\,|1|$ and $|-3-3| = 6 = 2\,|-3|$.

A circle with centre -1 and radius 2.



Key formulas

Distance: $|z-a| = \sqrt{(x-p)^2+(y-q)^2}$, $a = p+iq$

Read the point: $|z+1| = |z-(-1)|$

Circle: $|z-a| = r$, centre a, radius r

Inside / outside: $|z-a| < r$ / $|z-a| > r$

Line: $|z-a| = |z-b|$, the perpendicular bisector

Circle again: $|z-a| = k|z-b|$, $k>0$, $k \neq 1$

Diameter ends: $\dfrac{a+kb}{1+k}$ and $\dfrac{kb-a}{k-1}$

Number in front: $|2z+2| = 2\,|z+1|$

Practice set

Easy to hard. Read every modulus as a distance first, and mark the points: the picture tells you whether to expect a circle or a line. The answers start on the next page.

Q1. Find the centre and the radius of $|z - 3 + 2i| = 4$.

Q2. Find the centre of $|z + 1 - i| = 2$. Is 0 inside it?

Q3. Which of $3 + i$, $1 + 4i$ and $2 + 2i$ satisfy $|z - (1 + i)| < 2$?

Q4. Find the locus of z if $|z - 2 - 3i| = |z + 2 + 3i|$.

Q5. Find the locus of z if $|z - 4| = |z - 2i|$.

Q6. Find the locus of z if $|z + 2| = 2|z - 1|$.

Q7. Find the locus of z if $|z - i| = 3|z + i|$.

Q8. A student says $|z - 1| = |2z + 2|$ is a line, because it looks like $|z - a| = |z - b|$. Find the mistake and the correct locus.

Watch the animated lesson on YouTube

Locus of Complex Numbers: Circles and Lines

Scan the code to open the JEEVisionX channel on YouTube. This lesson is in the Complex Numbers playlist.

AN INITIATIVE BY AN IIT ALUMNUS

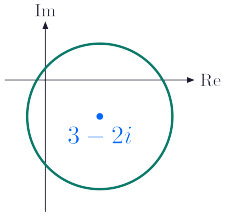
@JEEVisionX

Answers

Q1

Centre $3-2i$, radius 4

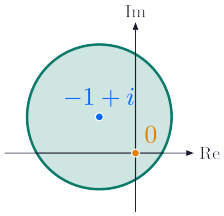
Write z minus a number: $z-3+2i=z-(3-2i)$. So the distance from $3-2i$ is always 4.



Q2

Centre $-1+i$; yes, 0 is inside

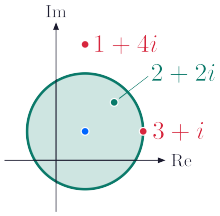
$z+1-i=z-(-1+i)$. The distance of 0 from the centre is $|0-(-1+i)|=\sqrt{2}$, less than the radius 2.



Q3

Only $2+2i$

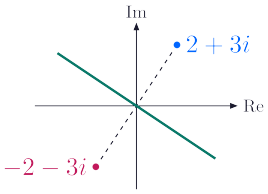
Distances from $1+i$: $3+i$ is 2 away (on the circle, not inside), $1+4i$ is 3 away (outside), $2+2i$ is $\sqrt{2}$ away (inside).



Q4

The line $2x+3y=0$

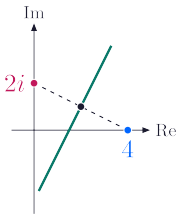
The perpendicular bisector of $2+3i$ and $-2-3i$. Squaring: $-4x-6y=4x+6y$, so $2x+3y=0$, through the midpoint 0.



Q5

The line $y=2x-3$

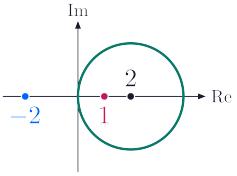
The perpendicular bisector of 4 and $2i$. Squaring: $-8x+16=-4y+4$. Check: it passes through the midpoint $2+i$.



Q6

A circle, centre 2, radius 2

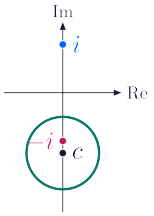
Squaring: $x^2+y^2-4x=0$. Quick way with $a=-2$, $b=1$, $k=2$: $P=0$ and $Q=4$, so the centre is 2 and the radius 2.



Q7

A circle, centre $-\frac{5}{4}i$, radius $\frac{3}{4}$

Squaring: $8x^2+8y^2+20y+8=0$. Quick way on the imaginary axis: $P=-\frac{1}{2}i$ and $Q=-2i$.



Q8

A circle, centre $-\frac{5}{3}$, radius $\frac{4}{3}$

$|2z+2|=2\,|z+1|$, so this is $|z-1|=2|z+1|$ with $k=2$, not 1: the circle of Apollonius from the lesson.

